ITERATIVE METHODS FOR SOLVING LINEAR SYSTEMS

*Practice 2. Solving symmetric sparse linear systems using SOR method
with Chebyshev's acceleration*

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OBJECTIVES

The purpose of this laboratory work is to implement the SOR method for solution of sparse linear systems and study ways to accelerate iterative methods by the example of the symmetric successive over relaxation.

ABSTRACT

The subject of this laboratory work is basic methods of solving linear systems with a symmetric positive definite matrix (Successive Over Relaxation and Symmetric Successive Over Relaxation) and a method to accelerate them based on Chebyshev's polynomials.

The test problem of this laboratory work is a linear system resulting from numerical solution of the Dirichlet problem for Poisson equation. In the course of work, it is proposed to implement the SOR method first and then apply one of the acceleration methods, i.e. Chebyshev's acceleration procedure. This acceleration procedure is applicable to the iterative methods whose iteration matrix has no complex eigenvalues. In general, the iterative matrix spectrum of the SOR method has complex values. However, this situation may be remedied using the Symmetric Successive Over Relaxation method. The iterative matrix of this method allows the use of Chebyshev's acceleration. For all the methods above experimental results are listed for a test problem.

GUIDELINES

The subject of this laboratory work is basic methods of solving linear systems (successive over relaxation and symmetric successive over relaxation) and a method to accelerate them based on Chebyshev's polynomials.

The test problem of this laboratory work is a linear system resulting from numerical solution of the Dirichlet problem for Poisson equation. Poisson equation is a partial differential equation for an unknown function. The finite difference method converts the equation to a linear system with a sparse symmetric positive definite matrix.

Within the scope of this laboratory work, it is proposed to implement the Successive Over-Relaxation (SOR) method for the purpose of solving $Ax=b$. This method is based on splitting the matrix A into three matrices $A=L+D+R$. Here, D is a $n \times n$ diagonal matrix whose principal diagonal coincides with that of the matrix A. L is the lower $n \times n$ triangular matrix whose non-zero (below-diagonal) elements also coincide with those of A and its principal diagonal is the zero one. Similarly, R is the upper $n \times n$ triangular matrix whose non-zero (above-diagonal) elements coincide with those of A and its principal diagonal is the zero one, too. The SOR method in the form of a matrix is formulated as $\dfrac{(D + \omega L)(x^{(s+1)} - x^{(s)})}{\omega} + Ax^{(s)} = b$.

In the first part of the laboratory work it is proposed to implement the SOR method. The laboratory work also features descriptions of the project, main and auxiliary functions. We will see the experimental results of the test problem to check the implemented algorithm for correctness.

One more method we propose to implement as part of this laboratory work is the SSOR, or Symmetric Successive Over Relaxation. As the name implies, this is a Successive Over Relaxation (SOR) method modification. One SSOR stage to calculate the $(s+1)^{th}$ approximation of $x^{(s+1)}$ consists of two steps:

- A SOR step that involves calculation of the intermediate approximation $x^{(s+1/2)}$ components in the normal order;

- A SOR step that involves calculation of the new approximation $x^{(s+1)}$ components in the reverse order.

This method demonstrates a lower convergence rate if compared to SOR, however, its advantage is the iteration matrix G symmetry which enables application of some acceleration methods. We will review the results of SSOR method application to the test problem solved earlier and compare them to those of the initial SOR method.

We will also implement the accelerated iterative process, i.e. the iterative method acceleration procedure based on Chebyshev's polynomials. The main idea of acceleration lies in the following.

Let the system $Ax=b$ be solved using any iterative method (e.g. SOR) with the iteration matrix G. If this process is convergent, i. e. $\rho(G) < 1$, the resulting sequence of vectors will converge to the true solution x^*. Now let us suppose that m iterations of the selected method took place, resulting in vectors $x^{(0)}, x^{(1)},..., x^{(m)}$, each being an approximation of x^*. It is known that one can find the linear combination $y^{(m)}$ of these vectors that will approximate x^* better than $x^{(m)}$. Solution of this problem is based on *Chebyshev's polynomials*. Using Chebyshev's polynomials, we may propose an effective method of coefficient computation for the linear combination $y^{(m)}$ without storing all vectors $x^{(0)}, x^{(1)},..., x^{(m)}$.

Unfortunately, the algorithm above is not directly applicable to the SOR method used for solving the $Ax=b$ system. The point is that the iterative process matrix G for the SOR method has, in general, complex eigenvalues, while Chebyshev's acceleration requires that the matrix G eigenvalues be real and belong to $[-\rho,\rho]$. However, this situation may be remedied using the SSOR method as its iteration matrix is symmetric.

For SSOR with Chebyshev's acceleration, computational experiments were performed to compare the resulting convergence rate with that of the initial method. As expected, this method

proved to be the best if used for solution of a test problem with known eigenvalues of the matrix *A* and iteration matrix *G*. Plus, a tenfold acceleration is observed if compared to SOR.

RECOMMENDATIONS FOR STUDENTS

SOR is a classical iterative method. See [1] for a detailed description of its features. A brief method description, including pseudocode algorithms, can be found in [2]. See a description of Chebyshev's acceleration in [3].

REFERENCES

1. David R. Kincaid and E. Ward Cheney. Numerical analysis : mathematics of scientific computing. Brooks/Cole Publishing Company, 1991.

2. J. Dongarra et al. Templates for the solution of linear systems: building blocks for iterative methods. SIAM, 1994.

3. James W. Demmel. Applied Numerical Linear Algebra. SIAM, 1997.

PRACTICE

1. Study the SOR method convergence rate depending on the method parameter.
2. Implement the Conjugate Gradient (CG) method and apply it to the test problem. Compare the number of iterations for the CG and SOR methods.
3. Apply Chebyshev's acceleration procedure to the conjugate gradient method. Compare the number of iterations for the initial and accelerated method.

TEST

1. Can the SOR method be considered as a direct method of solving linear systems?
   a. Yes
   b. Yes, but only for well-conditioned matrices.
   c. + No.

2. For a linear system with a SPD matrix, the SOR method will converge at
   a. $+\omega \in (0, 2)$
   b. $\omega \in (-2, 4)$
   c. any ω value

3. The SOR method can be effectively parallelized
   a. for small dense matrices
   b. +for large dense matrices
   c. for any dense matrices

4. SOR method
   a. Can be effectively parallelized for sparse matrices of any structure
   b. + Can be effectively parallelized for large block diagonal sparse matrices
   c. Cannot be effectively parallelized for sparse matrices
5. The SSOR method matrix eigenvalues are
   a. complex in general
   b. real within the range $[-\rho,\rho]$, where $\rho>1$
   c. +real within the range $[-\rho,\rho]$, where $0<\rho<1$
6. Chebyshev's acceleration may be applied to
   a. Any iterative method
   b. +An iterative method with a symmetric iteration matrix
   c. Any iterative method used to solve linear systems with a symmetric matrix
7. Chebyshev's acceleration may be applied if
   a. +the spectral radius of the method iteration matrix is estimated
   b. the spectral radius of the system matrix is estimated
   c. no spectral radius estimation is required.