Lobachevsky State University of Nizhni Novgorod

Faculty of Computational mathematics and cybernetics

Iterative Methods for Solving Linear Systems

Preconditioning Methods

Supported by Intel

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Problem statement

- Let us consider a system of n linear equations that looks as follows:

$$\begin{aligned} a_{11}x_1 &+ a_{12}x_2 &+ \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 &+ a_{22}x_2 &+ \ldots + a_{2n}x_n &= b_2 \\ \ldots & & & \\ a_{n1}x_1 &+ a_{n2}x_2 &+ \ldots + a_{nn}x_n &= b_n \end{aligned}$$

- This system can be represented as a matrix

$$Ax=b$$

- $A=(a_{ij})$ is a $n\times n$ real matrix; A is a sparse matrix; b and x are vectors consisting of n elements; the exact system solution is x^*.
- *An iterative method* generates a sequence of vectors $x^{(s)}\in R^m$, $s=0,1,2,\ldots$, where $x^{(s)}$ is an approximate system solution.

Convergence of Iterative Methods

- Iterative method is convergent if

$$\forall\ x^{(0)} \in R^m \quad \lim_{s\to\infty} \left\| x^{(s)} - x^* \right\| = 0$$

- For iterative methods, the following is true

$$\left\| z^{(s+1)} \right\| \le \left(\varphi(\mu_A)\right)^s \left\| z^{(0)} \right\|$$

where $z^{(s)}=x^{(s)}-x^*$ is the next approximation error,

φ is a function, $\varphi\to 0$ when $\sigma\to\infty$.

$\mu_A=\lambda_{max}/\lambda_{min}$ is the condition number.

Example for the conjugate gradient method

$$\varphi(\mu_A) = \frac{\sqrt{\mu_A} - 1}{\sqrt{\mu_A} + 1}$$

Idea of Preconditioning

- $\mu_A \approx 1$ – the convergence rate is high (A is well-conditioned)
- $\mu_A >> 1$ – the convergence rate is low(A is ill-conditioned)
- The idea of preconditioning lies in converting an ill-conditioned system

$$Ax=b$$

to a well-conditioned one

$$M^{-1}Ax=M^{-1}b.$$

Here, M is a preconditioner.
- $M^{-1}A$ is not computed explicitly as $M^{-1}A$ is very likely to be a dense matrix
- Corrective steps allowing for preconditioning are added to the iterative method.

Requirements to the Preconditioner

1. M must be close to A ($M^{-1}A$ is well-conditioned)
2. M must be easy to compute;
3. M must allow for fast solution of systems such as

$$Mz = r$$

in relation to an unknown vector z.

Example 1.

Let $M=A$. Then requirements 1 and 2 are satisfied.

Requirement 3 is not satisfied. $Az=r$ is the same problem as the initial one.

Example 2.

Let $M=\mathrm{diag}(A)$. Then requirements 2 and 3 are satisfied.

Requirement 1 may not be satisfied

Preconditioning Types

$Ax=b$ *is* the initial system

1. Left preconditioning
   $M^{-1}Ax=M^{-1}b.$
2. Right preconditioning
   $AM^{-1}u=b$, where $x=M^{-1}u.$
3. Split preconditioning
   Let us present the preconditioner as $M = M_L M_R$
4. Then
   $M_L^{-1} A M_R^{-1} u = M_L^{-1} b$, where $x = M_R^{-1} u$

Basic Preconditioners

- Let us remember the basic iterative methods that include Jacobi, Seidel and over relaxation (SOR and SSOR) methods.
- All the methods above are particular cases of the simple iteration method.

$$x^{(s+1)} = Gx^{(s)} + c$$

  where $A=M-N$ and $G = M^{-1}N = M^{-1}(M - A) = E - M^{-1}A$

- Simple iteration method (*) for the system

$$(E - G)x = c$$

  that may be formulated as follows

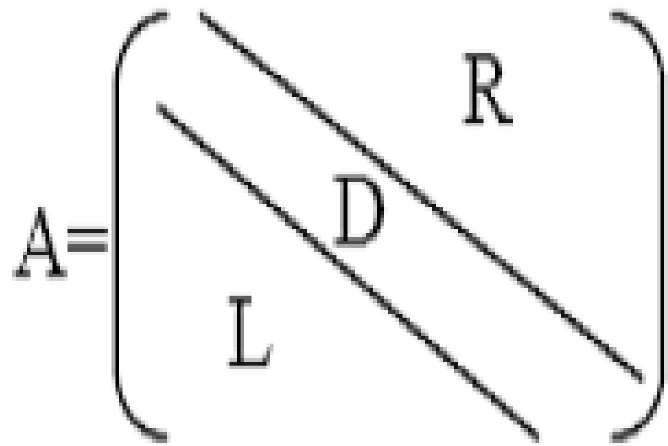
$$M^{-1}Ax = M^{-1}b$$

- As a result, the Jacobi, Seidel, SOR and SSOR methods are equivalent to the simple iteration method with a preconditioner.

Basic Preconditioners

Therefore, the following preconditioners are obtained:

1. $M_J = D$ (Jacobi)
   may be explicitly applied to the system - multiply by D^{-1}.
2. $M_{GS} = D + L$ (Gauss-Seidel)
3. $M_{SOR} = \frac{1}{\omega}(D + \omega L)$
4. $M_{SSOR} = \frac{1}{\omega(2-\omega)}(D + \omega L)D^{-1}(D + \omega R)$



SSOR-preconditioning

- How to select ω?
- Parameter selection for the preconditioner is not as critical as for the SSOR method: ω=1
- Symmetric Gauss-Seidel preconditioner

$$M_{SGS} = (D + L)D^{-1}(D + R)$$

- Use of preconditioner, i.e. system solution

$$M_{SGS} z = r$$

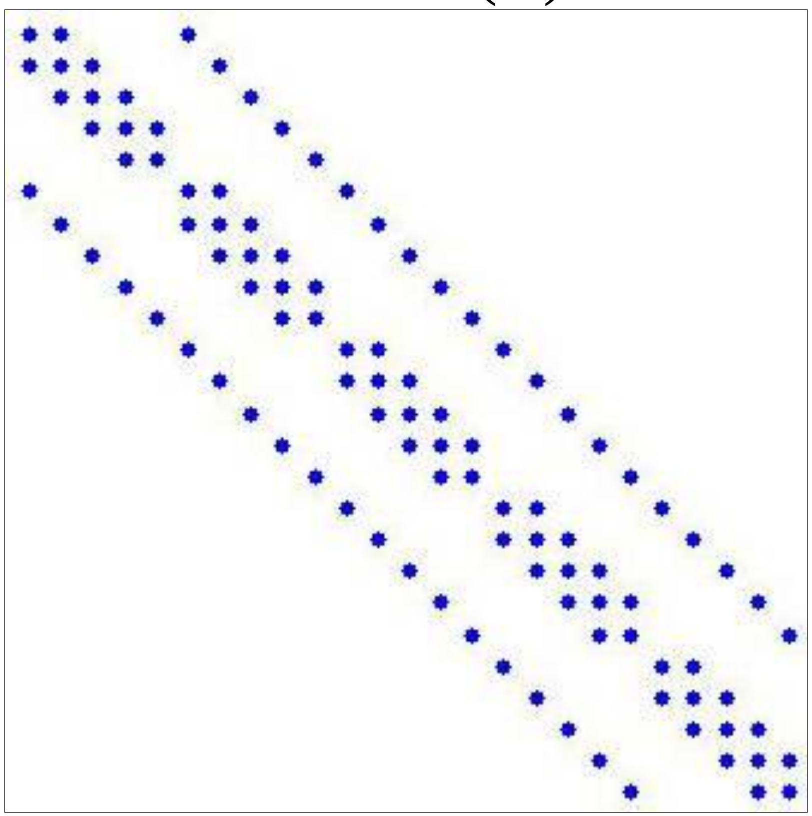
  is as complex as multiplying a matrix by vector.
- Upon the whole, M_{SGS} is better than M_J, but is still insufficient.

SGS-Preconditioning – Example

Numerical solution of Poisson Equation for a 5×5 grid

Matrix A: size n=25 (n^2 = 625), number of non-zeroes nz = 105,

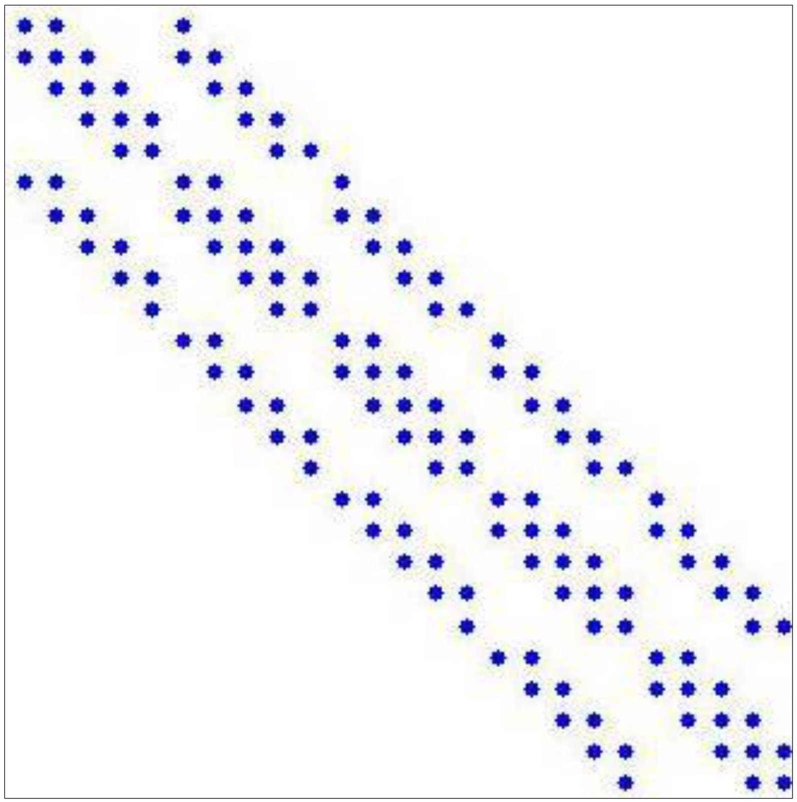
Condition number cond(A)= 20.7



SGS-Preconditioning – Example

Symmetric Gauss-Seidel preconditioner

$$M_{SGS} = (D + L) D^{-1} (D + R)$$



Initial system: cond(A)=20.7;
M_{SGS}: cond($M^{-1}A$)=5.1.

Poisson equation for a 40 × 40 grid
Matrix size 1600 × 1600.

Initial system:cond(A)=989;
M_{SGS}: cond($M^{-1}A$)=210.

ILU(0)-Preconditioning

- Let A be a sparse matrix
  $NZ(A)=\{(i,j): a_{ij}\neq 0\}$
- Let A factorization has been found in the form of

$$A=LU-R$$

  L and U are lower (with a single diagonal) and upper triangular matri ces;

  $NZ(L) \cup NZ(U) = NZ(A)$;

  $r_{ij} = 0$ for all $(i,j) \in NZ(A)$.

  Then ILU(0) is a preconditioner for $M=LU\approx A$.
- The requirements above do not provide for a unique determination of ILU(0).

ILU(0)-Preconditioning

- Constructive definition of ILU(0) : Perform LU–factorization of A zeroing all filling elements in L and U outside $NZ(A)$ at the same time.
- LU–factorization (method of Gaussian elimination)

for i=2,…, n do

for k=1,…, i–1 do

$a_{ik} = a_{ik}/a_{kk}$

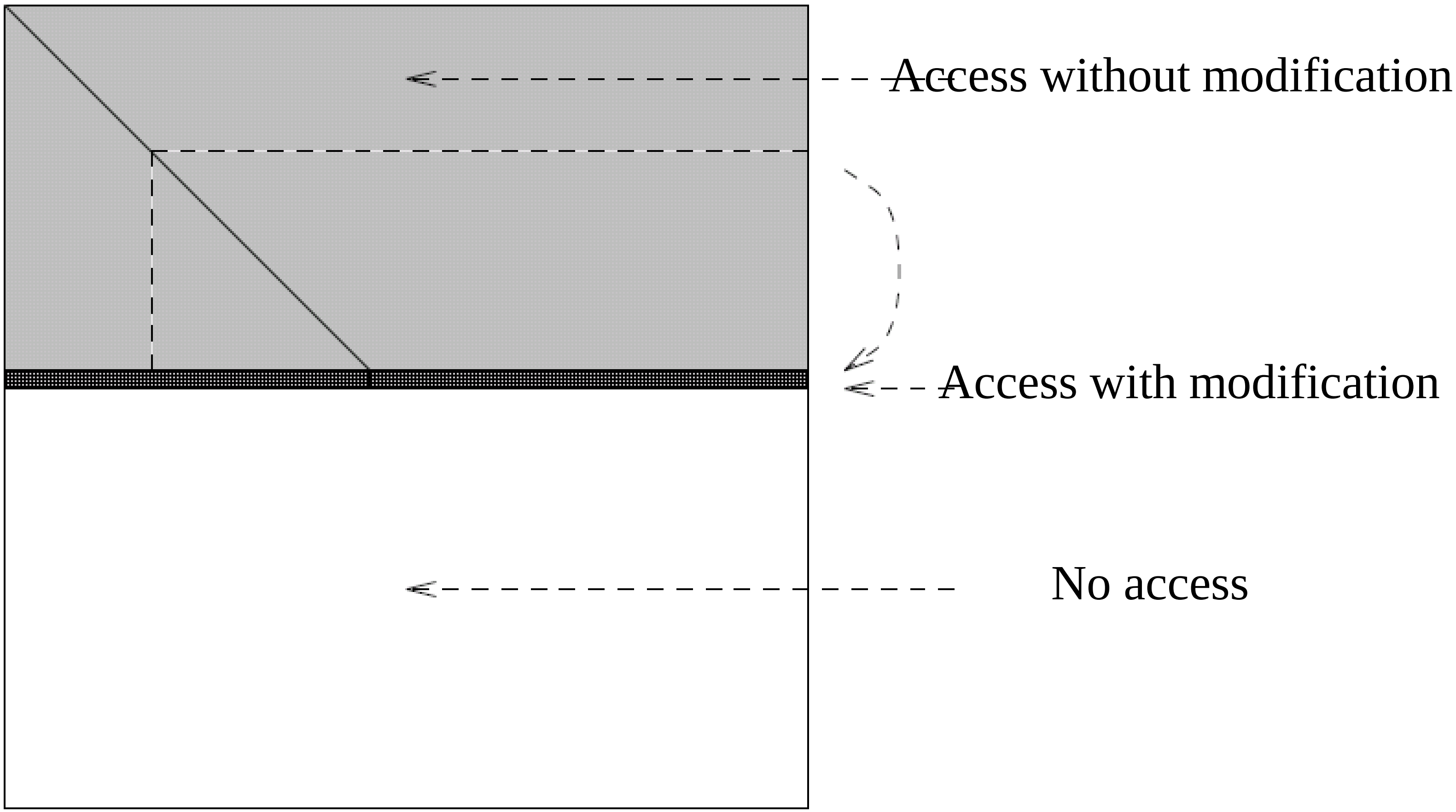
for j=k+1,…, n do

$a_{ij} = a_{ij} - a_{ik}*a_{kj}$

end i

end k

Memory State



Access to the matrix lines: effective for sparse matrices in CRS format

ILU(0)-Factorization - Algorithm

for i=2,…, n do

for k=1,…, i–1 and if $(i,k)\in NZ(A)$ do

$a_{ik} = a_{ik}/a_{kk}$

for j=k+1,…, n and if $(i,j)\in NZ(A)$ do

$a_{ij} = a_{ij} - a_{ik}*a_{kj}$

end i

end k

- If the matrix A is symmetric positive definite, then $ILU(0)$ transforms to $IC(0)$: this is incomplete Cholesky factorization.

ILU(0)-Factorization – Example 1

Let us consider factorization of the matrix A

$$A = \begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix}$$

and perform a complete LU-factorization

$$A = LU = \begin{bmatrix} 1 & & & \\ -0.25 & 1 & & \\ -0.25 & -0.067 & 1 & \\ 0 & -0.0267 & -0.286 & 1 \end{bmatrix} \begin{bmatrix} 4 & -1 & -1 & 0 \\ & 3.75 & -0.25 & -1 \\ & & 3.733 & -1.067 \\ & & & 3.429 \end{bmatrix}$$

ILU(0)-Factorization – Example 1

Incomplete factorization $A \approx IL * IU$

$$IL = \begin{bmatrix} 1 & & & \\ -0.25 & 1 & & \\ -0.25 & 0 & 1 & \\ 0 & -0.267 & -0.267 & 1 \end{bmatrix} \qquad IU = \begin{bmatrix} 4 & -1 & -1 & 0 \\ & 3.75 & 0 & -1 \\ & & 3.75 & -1 \\ & & & 3.467 \end{bmatrix}$$

Incomplete factorization residue

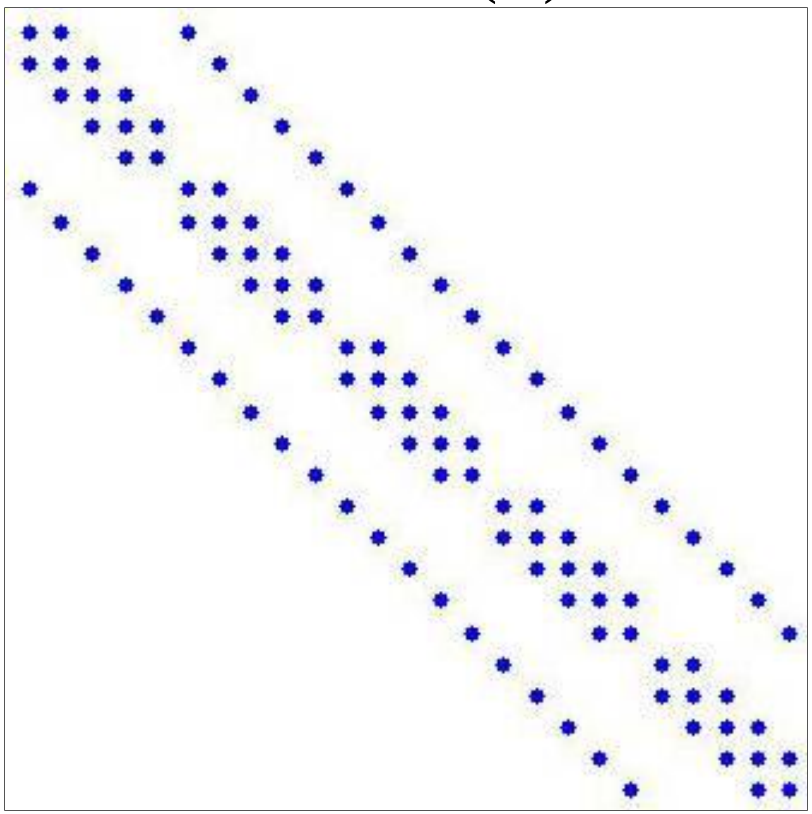
$$A - IL * IU = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -0.25 & 0 \\ 0 & -0.25 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

ILU(0)-factorization – Example 2

Numerical solution of Poisson equation for a 5×5 grid

Matrix A: size n=25 (n^2 = 625), number of nonzeroes nz = 105,

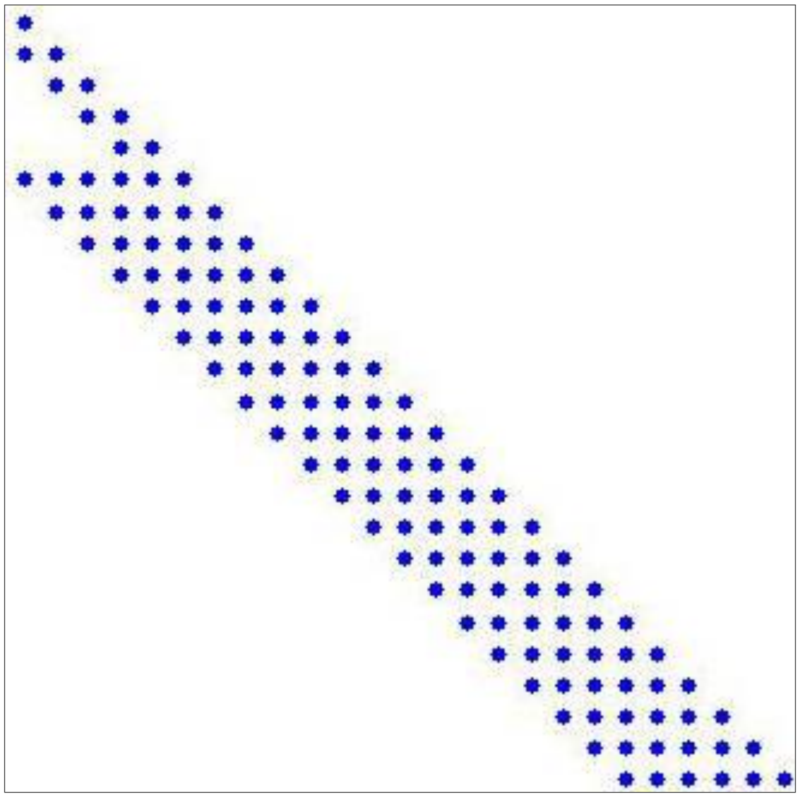
Condition number cond(A)= 20.7

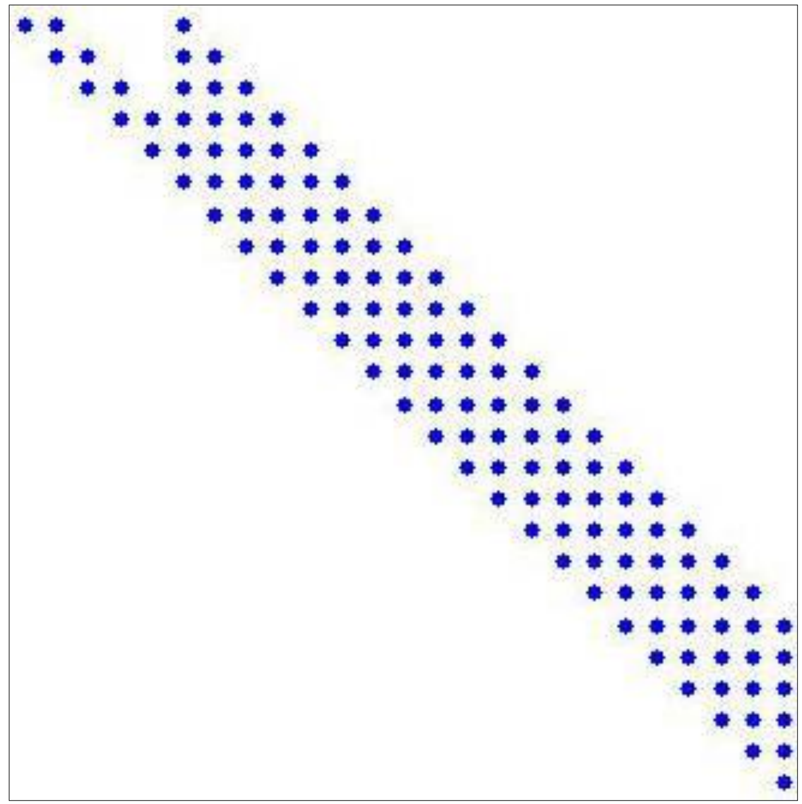


ILU(0)-Factorization – Example 2

Perform a complete LU-factorization

L U

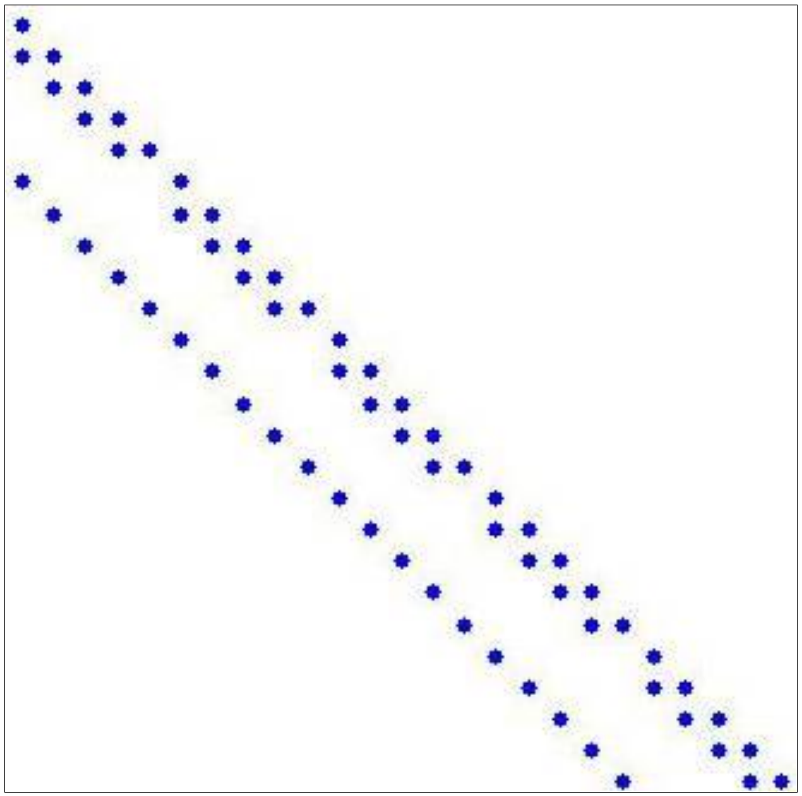


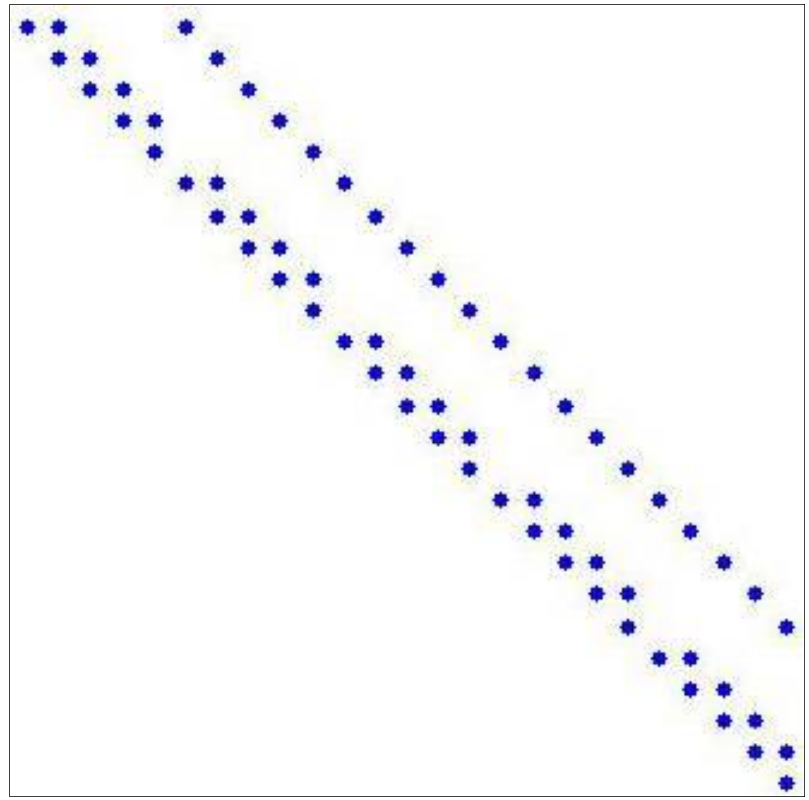


ILU(0)-Factorization – Example 2

Perform *ILU(0)*-factorization of $A \approx IL*IU$

IL *IU*

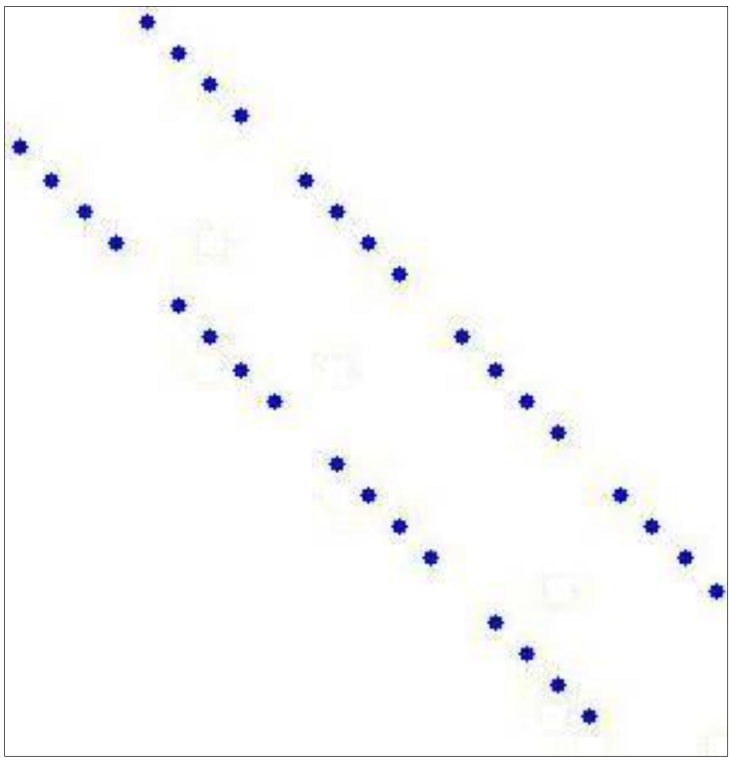




ILU(0)-Factorization – Example 2

ILU(0)-factorization residual

$$A - IL * IU$$



Initial system: cond(A)=20.7;

M_{SGS}: cond($M^{-1}A$)=5.1.

$M_{ILU(0)}$: cond($M^{-1}A$)=3.6.

Poisson equation for a 40 × 40 grid
Matrix size 1600 × 1600.

Initial system:cond(A)=989;

M_{SGS}: cond($M^{-1}A$)=210.

$M_{ILU(0)}$: cond($M^{-1}A$)=143.

Filling Control. ILU(p)-Factorization

- A more exact ILU-factorization can be obtained by allowing a certain degree of factor filling
  - for matrices with a regular structure *p* additional diagonals may be filled;
  - generalization for matrices with an irregular structure via the *filling level* concept.
- Initial l_{ij} filling level value

$$l_{ij} = \begin{cases} 0, \text{если } a_{ij} \neq 0 \text{ или } i = j, \\ \infty, \text{иначе.} \end{cases}$$

- At the *i*-th step of Gaussian elimination

$$l_{ij} = \min\{ l_{ij}, l_{ik} + l_{kj} + 1\}$$

ILU(p)-Factorization - Algorithm

- ILU(p) strategy is to zero all elements whose filling level exceeds *p*.

for $i = 2,\ldots,n$ do

  for $k = 1, \ldots, i-1$ and if $a_{ij} \neq 0$ do

    $a_{ik} = a_{ik} / a_{jj}$

    $a_{i*} = a_{i*} - a_{ik} * a_{i*}$

    update filling levels for a_{i*} :

    for *i*-th line: if $l_{ij} > p$ then $a_{ij} = 0$

$$l_{ij} = \min\{ l_{ij}, l_{ik} + l_{kj} + 1\}$$

  end *k*

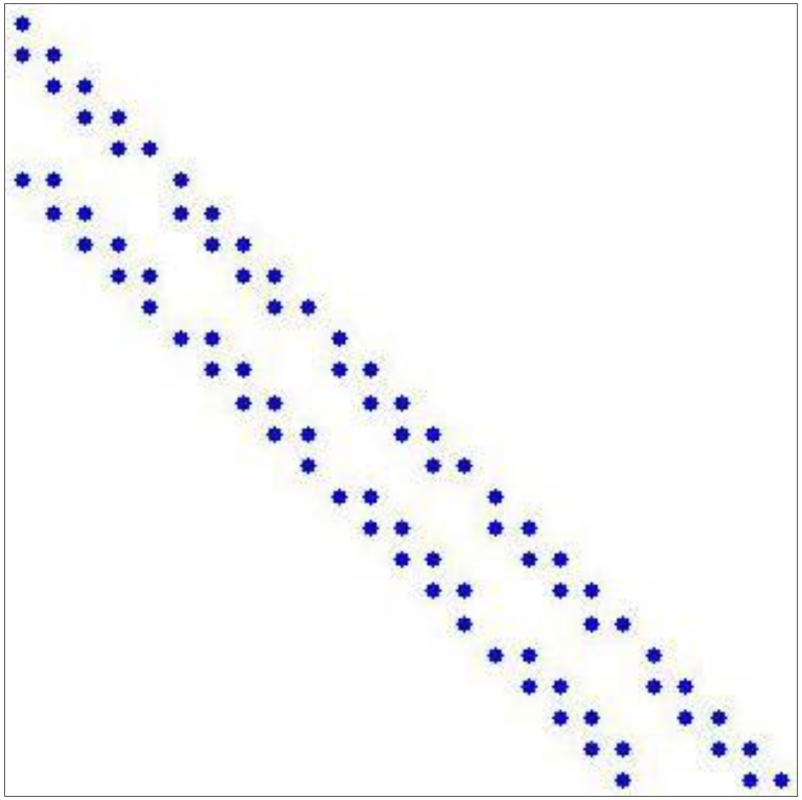
end *i*

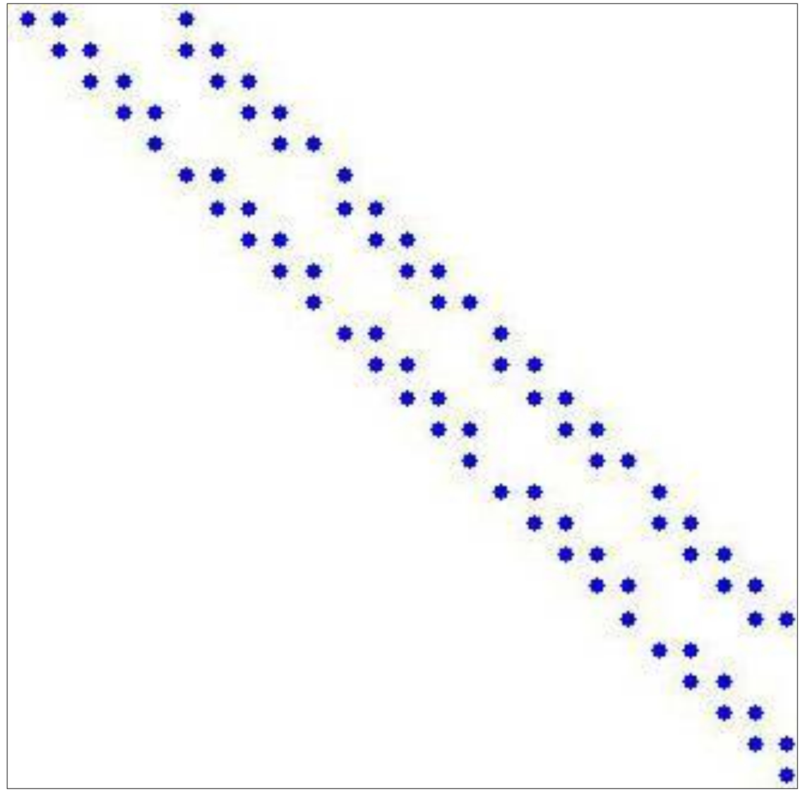
- The algorithm may be divided into two parts: symbolic (*L* and *U* patterns) and numeric (*L* and *U* values)

ILU(p)-Factorization - Example

Numerical solution of Poisson equation for a 5×5 grid

Perform *ILU(1)*-factorization of $A \approx IL * IU$

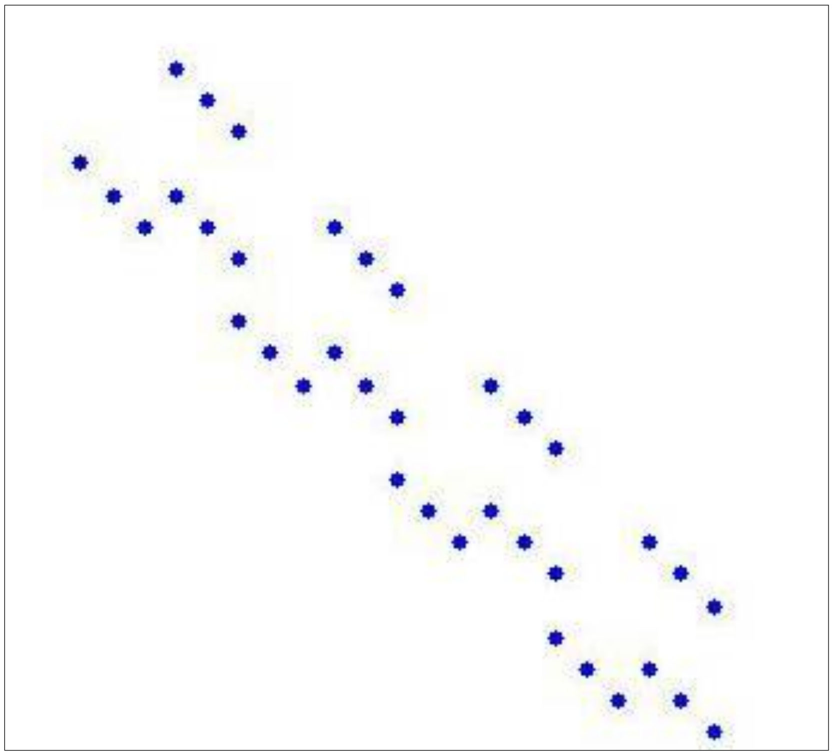
IL 

IU 

ILU(p)-Factorization - Example

ILU(1)-factorization residual

$A – IL * IU$



Initial system: cond(A)=20.7;
M_{SGS}: cond($M^{-1}A$)=5.1.
$M_{ILU(0)}$: cond($M^{-1}A$)=3.6.
$M_{ILU(1)}$: cond($M^{-1}A$)=1.5.

Poisson Equation for a 40 × 40 grid
Matrix size 1600 × 1600.

Initial system:cond(A)=989;
M_{SGS}: cond($M^{-1}A$)=210.
$M_{ILU(0)}$: cond($M^{-1}A$)=143.
$M_{ILU(0)}$: cond($M^{-1}A$)=54.

Conclusion

- As part of this lecture, we have reviewed the following:
  - Concept of preconditioning
  - Requirements to preconditioners
  - Preconditioning types
  - Basic preconditioners
    - Jacobi (J), Gauss-Seidel (GS)
    - SOR, SSOR, SGS
  - Partial LU-factorization
    - General Pattern
    - ILU(0), factorization without filling
    - ILU(0), factorization with filling control
  - Experimental results

References

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Internet Resources

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