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NUMERICAL METHODS FOR SOLVING DIFFERENTIAL EQUATIONS

Lecture 2. Numerical Methods for Solving Systems of Stochastic Differential Equations

Nizhni Novgorod
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OBJECTIVES

The purpose of this lecture is to study a number of explicit methods for solving stochastic differential equations.

ABSTRACT

This lecture makes an introduction to the methods for SDE numerical integration illustrated by financial market modeling. The lecture also gives background information about SDE and the respective classical explicit solutions methods, brings up for discussion the issues of method convergence and correctness of numerical modeling results.

GUIDELINES

A stochastic differential equation (SDE) is a complex mathematical object has numerous applications in science and engineering (namely in statistical mechanics, electrodynamics, quantitative finance, management theory, chemistry, biology, economics etc [1, 2]). SDEs and their systems enable description of dynamical systems behaviour with allowance for random factors.

Let us consider a vectorized system of d stochastic differential equations (SDE): $\mathrm{dX_t} = \mathrm{a(X_t, t)dt + b(X_t, t)dW_t}$, where $t \geq 0$ is the time, $X_t \stackrel{\text{def}}{=} X(t)$ is the sought vector value, a d-dimensional continuous time random process, $X(0) = X_0 \in R^d$ is the initial condition, $a : R^d \times [0; \infty) \to R^d$, b is the matrix function $b : R^d \times [0; \infty) \to R^{d\times m}$, $W = \{W_t = (W_t^1, W_t^2, \ldots, W_t^m)^T,\ t \geq 0\}$ is the m-dimensional standard Wiener process with components $W_t^1, W_t^2, \ldots, W_t^m$. The difference of this system from an ODE is that its right-hand part has an additional summand that contains the multiplier dW_t (a random Wiener process differential) which, in its turn, describes the influence of random factors on the dynamics of $\mathrm{X_t}$. The Wiener process is the mathematical model of continuous-time Brownian motion. It is often used for random influence modeling.

The lecture also features a quantitative finance problem. Let consider a financial market of two types of assets, i. e. stocks (risk-generating assets, *S*) and bonds (risk-free assets, *B*), evolve in continuous time. For market modeling, the well-known Black-Scholes model will be used. The equation $dB_t = rB_t dt, \quad B_0 > 0$ is an ordinary differential equation that describes behaviour of the bond B_t price depending on the interest rate r. The equation $dS_t = S_t(rdt + \sigma dW_t), \quad S_0 > 0$ is a stochastic differential equation that describes evolution of the stock price S_t depending on the interest rate r, volatility σ and the Wiener process $W = (W_t)_{\{t \geq 0\}}$. Initial prices of the stock and bond (S_0 and B_0, respectively) are set values. The work gives an explanation of the economic substance of the model. Given invariable market parameters and a number of other conditions, the second equation of the Black-Scholes model will have an analytical solution: $S_t = S_0 e^{\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t}$.

The issue of possible use of known numerical methods for integration of ODE into SDE has received a detailed discussion in the literature which proved that this approach lacks efficiency. This work features three well-known methods based on the stochastic equivalent of the Taylor series. The lecture gives definitions of convergence, strong and weak convergence of the methods for solving SDEs. From the point of view of intensionality, strong convergence means that for any sufficiently small integration step h the average error at the end point $t = T$ is bounded from above by the step value to the power γ multiplied by a certain invariable independent of h. The greater is the γ value, the greater is the convergence rate when the step h is reduced. Thus, if the method convergence order is 1, if the step is halved, the error in the point $t = T$ is also expected to be halved.

The work also gives the formula for the modified Euler method (the Euler-Maruyama method) of solving stochastic differential equations. The recursive formula of solving SDEs looks as follows:

$$X_{i+1} = X_i + A(X_i, t_i)h + B(X_i, t_i)\Delta W_i, \qquad X_0 = x_0.$$

It is known that the order of strong convergence for this method is ½.

The Milstein method is based on the stochastic equivalent of the Taylor series, uses one summand more than the previous method and has the following formula:

$$X_{i+1} = X_i + A(X_i, t_i)h + B(X_i, t_i)\Delta W_i + \frac{1}{2} B(X_i, t_i) B'_X(X_i, t_i)((\Delta W_i)^2 - h), \ \ X_0 = x_0.$$

It is known that the order of strong convergence for this method is 1.

The Burrage-Platen method also makes allowance for additional members of the Taylor series stochastic equivalents. Its computational formulas contain the random process increment Z_t that has Gaussian random variable with zero mean and variation $\frac{1}{3}h^3$ correlated to ΔW_i. It is known that the order of strong convergence for this method is 1.5.

To confirm that the numerical solution $Y(T)$ converges to the sought function $X(T)$ one has to make sure that experimental data satisfy the relation $E\{|X(T) - Y(T)|\} \leq Ch^{\gamma}$. Taking the logarithm of the computation formula of ε will result in a straight-line equation of the form $Y = \gamma X + b$, where γ determines the strong convergence order for the numerical method.

RECOMMENDATIONS FOR STUDENTS

Monographs [1, 2, 3] feature a detailed formalized description of stochastic differential equations and random processes. This work is mostly based on materials [54] and monograph [2].

REFERENCES

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3. Oksendal B.K. Stochastic Differential Equations: An Introduction with Applications. – Berlin: Springer, 2003.
4. Schafter T. Numerical Integration of SDEs: A Short Tutorial, Swiss Federal Institute of Technology in Lausanne (EPFL), Switzerland, 2010. – Unpublished manuscript. – [http://infoscience.epfl.ch/record/143450/files/sde_tutorial.pdf]
5. Higham D.J. An Algorithmic Introduction to Numerical Simulation of Stochastic Differential Equations // SIAM review, Vol. 43, No 3. – pp. 525–546.

PRACTICE

1. Implement the Euler, Milstein and Burrage-Platen methods to solve SDEs for the purposes of finance market modeling. Check computational experiments for compliance with the theoretical method convergence orders.
2. Study other methods of SDE numerical integration, make their software implementation and analyze the convergence order.

TEST

1. What is modeled by the Wiener process?
    a. Discrete-time Brownian motion
    b. + Continuous-time Brownian motion
    c. Particle distribution in the vacuum
2. What distribution does the Wiener process value $W_t - W_s$, $s < t$ satisfy?
    a. + Normal
    b. Uniform
    c. Exponential
3. What does volatility σ in the Black-Scholes method characterize?
    a. + Market process randomness degree
    b. Stock price modification rate
    c. Stock price growth rate
4. Indicate analytic solution of the Black-Scholes equation
    a. $S_t = S_0 e^{(r-\sigma^2)t+\sigma W_t}$
    b. $S_t = S_0 e^{\left(r-\frac{\sigma^2}{2}\right)t-\sigma W_t}$
    c. + $S_t = S_0 e^{\left(r-\frac{\sigma^2}{2}\right)t+\sigma W_t}$
5. What is the difference between the formulas of the Euler-Maruyama method and the Milstein method of solving SDEs?
    a. Allowance for correlation between the values
    b. Presence of a random Wiener process
    c. +Allowance for an additional Taylor series summand
6. What is the strong convergence order of the Milstein method?
    a. ½
    b. + 1
    c. 1½
7. What is the strong convergence order of the Burrage-Platen method?
    a. ½
    b. 1
    c. + 1½
8. What are the random process Z_t parameters from the Burrage-Platen method computation formula?
    a. + Average 0, variance $\frac{1}{3}h^3$, $E(\Delta W_i \Delta Z_i) = \frac{1}{2}h^2$.
    b. Average h, variance $\frac{1}{3}h^3$, $E(\Delta W_i \Delta Z_i) = \frac{1}{2}h^2$.
    c. Average 0, variance $\frac{1}{2}h^2$, $E(\Delta W_i \Delta Z_i) = \frac{1}{3}h^3$.
9. Choose the correct definition of a numerical method with the strong convergence order γ
    a. + The numerical method converges to $X(t)$ at the time T with the order $\gamma > 0$ if there is an invariable $C > 0$ independent of h and the value $\delta > 0$ so that $E\{|X(T) - Y(T)|\} \leq Ch^\gamma$ for all$h \in (0; \delta)$.
    b. The numerical method converges strongly to$X(t)$at the time$T$ with the order$\gamma > 0$ if $\lim_{h\to 0} E\{|X(T) - Y(T)|\} = 0$, where E is the expectation.
    c. The numerical method converges strongly to$X(t)$ at the time$T$ with the order$\gamma > 0$ if $\lim_{h\to 0} E\{|X(T) - Y(T)|\} = h^\gamma$ where E is the expectation.

10. What value is used to check convergence of the numerical solution$Y(t)$ to the solution of the SDE $X(t)$ at the time T?
    a. Variance $D\{|X(T) - Y(T)|\}$
    b. Correlation $E(X(T), Y(T))$
    c. + Expectation $E\{|X(T) - Y(T)|\}$

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