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## **NUMERICAL METHODS FOR SOLVING DIFFERENTIAL EQUATIONS**

*Practice 2. Methods for Solving Systems of Stochastic Differential Equations as  
Illustrated by Financial Market Modeling*

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## OBJECTIVES

The purpose of this work is to implement explicit methods of numerical solution of stochastic differential equations and to study the numerical solution convergence to the SDE solution.

## ABSTRACT

This work consists in implementing the numerical solution of a SDE describing a stock market. It involves numerical modeling of the Wiener process using the pseudo-random generator from the Intel MKL library. This is followed by implementation of the Euler, Milstein and Burrage-Platen methods and subsequent evaluation of the numerical solution error.

## GUIDELINES

This practical work features a quantitative finance problem. Let a financial market of two types of assets, i. e. stocks (risk-generating assets,  $S$ ) and bonds (risk-free assets,  $B$ ), evolve in continuous time. For market modeling, the well-known Black-Scholes model will be used. The equation  $dB_t = rB_t dt$ ,  $B_0 > 0$  is an ordinary differential equation that describes behaviour of the bond  $B_t$  price depending on the interest rate  $r$ . The equation  $dS_t = S_t(rdt + \sigma dW_t)$ ,  $S_0 > 0$  is a stochastic differential equation that describes evolution of the stock price  $S_t$  depending on the interest rate  $r$ , volatility  $\sigma$  and the Wiener process  $W = (W_t)_{t \geq 0}$ . Initial prices of the stock and bond ( $S_0$  and  $B_0$ , respectively) are set values.

Now let us discuss numerical modeling of the Wiener process paths. Use a computational scheme that lets obtain **nPaths** paths of the random process  $W_t$ . To do this, set the number of steps  $nStep$ , the time step  $h$  and compute for each path  $i$  at each moment of time  $j$   $W[i, j] = W[i, j-1] + dW$ , where  $dW$  is a normally distributed random value. In such a way, obtain **nPaths** paths of the random process  $W_t$  which are sets of discrete values at the point  $nSteps+1$  of the segment  $[0;T]$ .

Then perform numerical modeling of the Wiener process using the pseudo-random (PRN) generator from the Intel MKL library. To obtain pseudorandom numbers with standard normal distribution, declare a service variable of type **VSLStreamStatePtr**, i. e. a PRN stream, and call the following functions:

1. **vslNewStreamEx()** is the basic generator initialization. The function inputs are the pointer to the stream variable and the basic generator type. The initial value to initialize the generator for this function is composed from **k** 32-bit numbers stored in the array. These are also input parameters of the function;
2. **vdRngGaussian()** is generation of Gaussian distribution. The function inputs are the constant that determines the algorithm of random distribution conversion to normal distribution (**VSL\_METHOD\_DGAUSSIAN\_ICDF** in our case), stream variable, size of buffer for the returned PRNs, buffer itself and distribution parameters (expectation and standard deviation);
3. **vslDeleteStream()** release the random stream.

The basic generator type used is **VSL\_BRNG\_MCG59**. Generator initialization is determined by the **const unsigned int seed[2]** array whose elements construct one 64-bit number. The buffer size of the PRN generation function influences only the efficiency. When **vdRngGaussian()** is called for the next time with the same parameters, we shall obtain the next block whose numbers follow the same sequence as the one defined by **seed**. To check

generation results, output the resulting random numbers to the file and find expectation and standard deviation for them.

Given constant market parameters and a number of other conditions, the second equation of the Black-Scholes model will have an analytical solution:  $S_t = S_0 e^{\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t}$ . To forecast the stock price at the time  $T$ , it is proposed to use the Monte Carlo algorithm, i. e. to generate the  $W_T$  value **nPaths** times, use it in the formula and average the results. Create an implementation of this algorithm. Then it is proposed to modify it: compute the  $S(t)$  values not only in the endpoint of  $[0; T]$ , but also in its midpoints. For this purpose, divide the interval into  $nSteps$  parts with an equal step  $h$  and use the Wiener process generator implemented earlier.

Now let us study computational formulas of SDE numerical solution methods, i. e. the Euler, Milstein and Burrage-Platen methods, evaluate their convergence and create implementations of these methods. Implementation of the Euler and Milstein methods will require calling the step calculation function. The Burrage-Platen method will require modification of the Wiener process path modeling function to model the  $Z_t$  process. To generate multivariate normal distribution of  $Z_t$  and  $W_t$ , **vdRngGaussianMV()** will be used. The inputs for this function are the RN generation method name, RN stream, dimension of random vectors, size of buffer to save the returned PRNs, buffer itself, covariance matrix storage format, expectation and covariance matrix. Calling for the function above will results in **nSteps** random vectors sized 2.

To confirm that the numerical solution  $Y(T)$  converges to the sought function  $X(T)$  one has to make sure that experimental data satisfy the relation  $E\{|X(T) - Y(T)|\} \leq Ch^\gamma$ . For **nPaths** paths of the Wiener process  $W_t$  compute  $\varepsilon = \frac{1}{nPaths} \sum_{k=1}^{nPaths} \{|X^k(T) - Y^k(T)|\}$  to be used to evaluate expectation  $E\{|X(T) - Y(T)|\}$ . The complete set of paths can be divided into  $M$  groups, compute absolute variance for the exact value for each path, then compute  $M$  averages and, using them as a sampling, find the sample mean and variance. Using the parameters above, one can construct an additional interval for  $\varepsilon$ . Taking the logarithm of the computation formula for  $\varepsilon$  will result in a straight-line equation of the form  $Y = \gamma X + b$ , where  $\gamma$  determines the strong convergence order for the numerical method.

The  $S_t$  stock price modeling error can be evaluated as follows: perform numerical modeling of paths for the Wiener process  $W_t$  and for the Burrage-Platen method process  $Z_t$  followed by analytical and numerical SDE solution and  $\varepsilon$  computation for each path. It is proposed to perform computations several times using a different integration step and represent the results as a table. After this, take logarithms of the results, represent the data as a table function with  $X$  and  $Y$  columns and, using the least squares method, evaluate the  $\gamma$  value and the least squares method error. A negligible least squares method error and the  $\gamma$  value corresponding to the theoretical convergence order will indicate that the experimental results of the implementation do not clash with the theory.

The experimental results mentioned in this work prove correctness of the effected implementation.

## RECOMMENDATIONS FOR STUDENTS

See the monographs below [2, 3, 4] for a detailed formalized description of stochastic differential equations and random processes. This work is mostly based on materials [1], [5] and monograph [3].

## REFERENCES

1. Higham D.J. An Algorithmic Introduction to Numerical Simulation of Stochastic Differential Equations // SIAM review, Vol. 43, No 3. – pp. 525–546.

2. Kloeden P. E., Platen E., Schurz H. Numerical solution of SDEs through computer experiments. – Berlin: Springer, 1997.
3. Kloeden P.E., Platen E. Numerical solution of stochastic differential equations. – Berlin: Springer, 1992.
4. Oksendal B.K. Stochastic Differential Equations: An Introduction with Applications. – Berlin: Springer, 2003.
5. Schafer T. Numerical Integration of SDEs: A Short Tutorial, Swiss Federal Institute of Technology in Lausanne (EPFL), Switzerland, 2010. – Unpublished manuscript. – [[http://infoscience.epfl.ch/record/143450/files/sde\\_tutorial.pdf](http://infoscience.epfl.ch/record/143450/files/sde_tutorial.pdf)]

## PRACTICE

1. Implement parallel versions of the studied methods.
2. Implement a generator of random numbers subject to normal distribution using the Box-Muller algorithm. Check the implementation for correctness. Replace the PRNG in the financial market modeling program, perform numerical experiments and compare them to the results obtained earlier.

## TEST

1. What MKL file is to be connected to enable the use of PRNGs?
  - a. mkl\_blas.h
  - b. mkl\_lapack.h
  - c. + mkl\_vsl.h
2. What is the input parameter for `vdRngGaussian()`?
  - a. + Standard deviation
  - b. Covariance matrix
  - c. Covariance matrix storage format
3. What is returned by `vdRngGaussian()`?
  - a. PRN stream
  - b. Pointer to the buffer containing PRNs
  - c. + Error code
4. What is NOT an input parameters for `vdRngGaussianMV()`?
  - a. + Standard deviation
  - b. Covariance matrix
  - c. PRN generation method
5. What correlation matrix storage format does the `VSL_MATRIX_STORAGE_PACKED` constant correspond to?
  - a. Column Row Storage
  - b. + The lower matrix triangle is recoded column-wise to a single-dimension array
  - c. The lower matrix triangle is recoded row-wise to a single-dimension array
6. What condition must be met by the numerical modeling data in case of strong convergence of the result  $Y(t)$  to the analytical solution  $X(t)$  with the order  $\gamma$  at time  $T$ ?
  - a.  $E\{|X(T) - Y(T)|\} \leq Ch^\gamma$
  - b.  $E\{|X(T) - Y(T)|\} \leq Ch\gamma$
  - c.  $E\{|X(T) - Y(T)|\} \leq C\gamma^h$

7. What linear equation corresponds to the condition of strong convergence of the result  $Y(t)$  to the analytical solution  $X(t)$  with the order  $\gamma$  at time  $T$ ?
  - a.  $X = \gamma Y + b$
  - b.  $Y = X + \gamma$
  - c.  $+ Y = \gamma X + b$
8. What part of the random sequence will be computed by the PRN generator **vdRngGaussian()** when it is recalled, if generator initialization was called only once?
  - a. + Next block of numbers from the same sequence
  - b. Same random numbers as for the first time
  - c. Returned numbers do not depend on the number of times the function was called.
9. What condition is necessary to enable analytical solution of SDE in the Black-Scholes model?
  - a. Zero  $\sigma$  volatility
  - b. + Constant market parameters  $r, \sigma$
  - c. Zero correlation between the market parameters  $r, \sigma$
10. What advantages does the stepwise Wiener process generation within the  $[0, T]$  interval and subsequent value averaging have compared to single  $W_T$  value generation?
  - a. Generation time reduction
  - b. + More clear definition of the generated number expectation
  - c. Reduction of variance for generation