Introduction to Bilevel Programming

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Outline

Bilevel programming: goal and motivation

Goal of this course

To be able to distinguish **bilevel problems** from **other optimization problems** with **multiple decision makers** and/or **multiple decision levels**.

- Standard **Mathematical Programming (MP)** models are often **inadequate** in the real-life because more than a single objective and one **Decision Maker (DM)** are involved.
- **Multi-Objective Programming (MOP)** deals with the **extension** of optimization techniques to account **several objective** functions.
- **Game Theory (GT)** deals with the mathematical models of **conflict and cooperation** between intelligent rational **decision makers**

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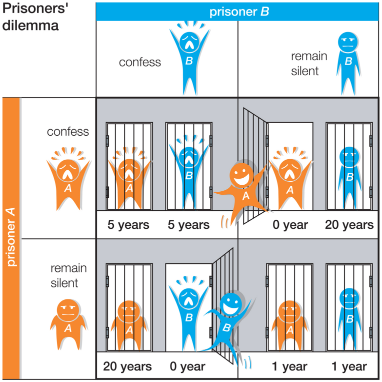

Bilevel Programming (BP) in a narrow sense is the combination of both.

Bilevel Programming Applications

The world is multilevel!

Applications of Bilevel Programming are diverse and include:

- Parameter Estimation [Mitsos et al., 2008]
- Management of Multi-Divisional Firms [Ryu et al., 2004]
- Environmental Policies: Biofuel Production [Bard et al., 2000]
- Traffic Planning [Migdalas, 1995]
- Chemical Equilibria [Clark, 1990]
- Design of Transportation Networks [LeBlanc and Boyce, 1985]
- Agricultural Planning [Fortuny-Amat and McCarl, 1981]
- Optimisation of Strategic Defence [Bracken and McGill, 1974]
- Resource Allocation [Cassidy et al., 1971]
- Stackelberg Games: Market Economy [Stackelberg, 1934]

Bilevel Programming Applications

Seller-Buyer Strategies

Seller-Buyer Strategies

- An **owner** of a company (**leader**) dictates the selling price and supply. He wants to **maximize** his **profit**.
- The **buyers** (**followers**) look at the product quality, pricing and various other options available to **maximize** their **performance**.

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Bilevel formulation

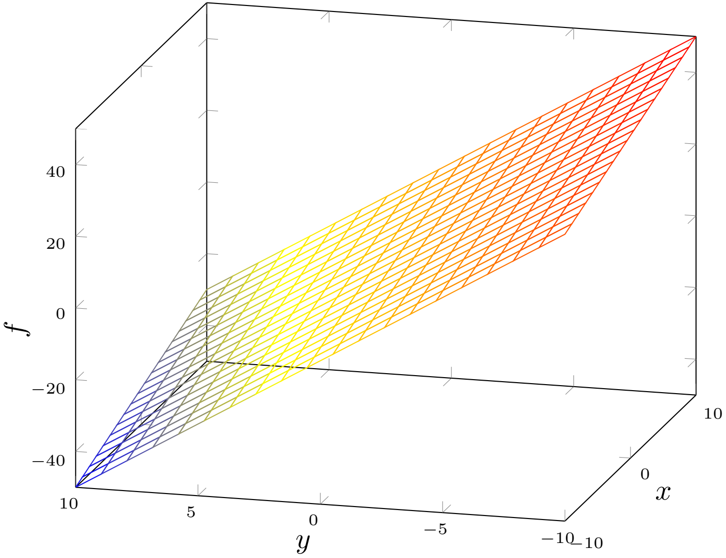
max profit (selling price, supply, demand, other variables)
s.t. upper level constraints
    max performance (selling price, quality, demand)
    s.t. lower level constraints

Single-objective optimization problem

Single objective function

Example 1

$$\begin{aligned} \min_{x,y} \; f(x,y) &= x - 4y \\ \text{s.t.} -x - y &\leq -3 \\ -2x + y &\leq 0 \\ 2x + y &\leq 12 \\ 3x - 2y &\leq 4 \\ x \geq 0 \quad & \quad y \geq 0 \end{aligned}$$



Single-objective optimization problem

Constraints

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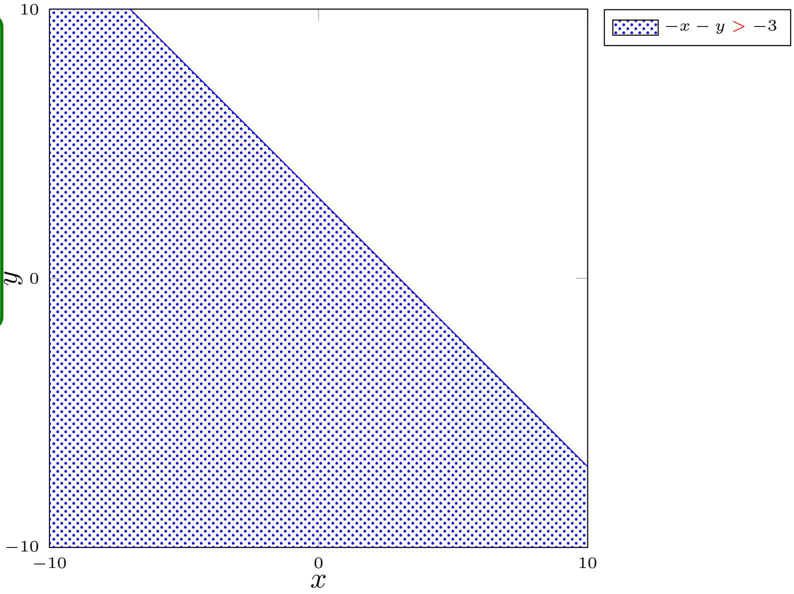

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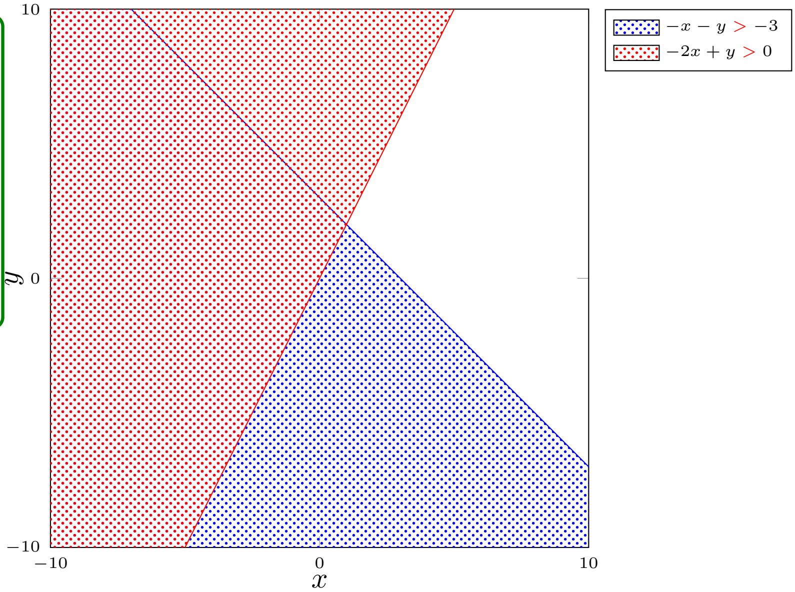

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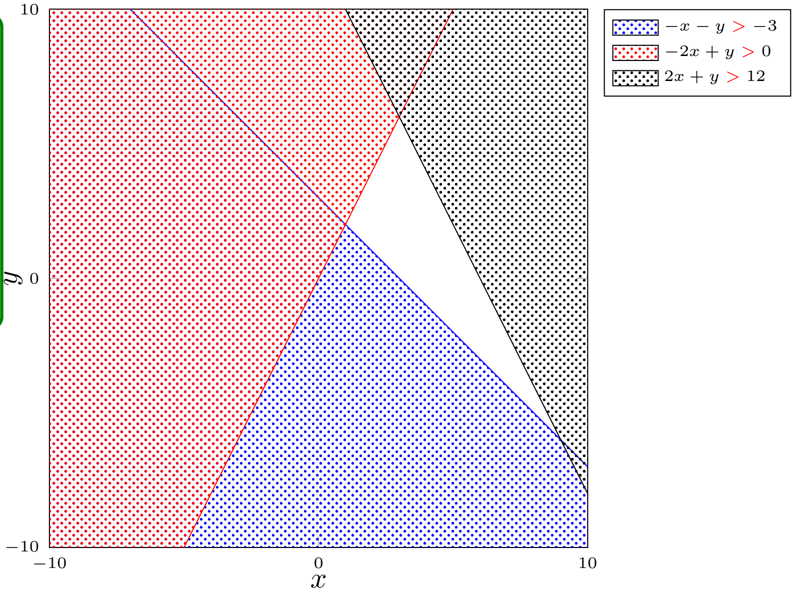

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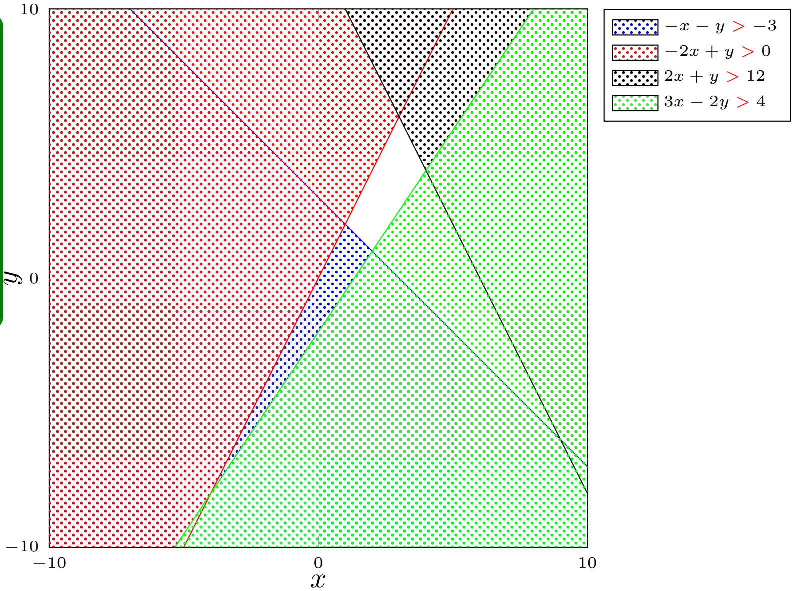

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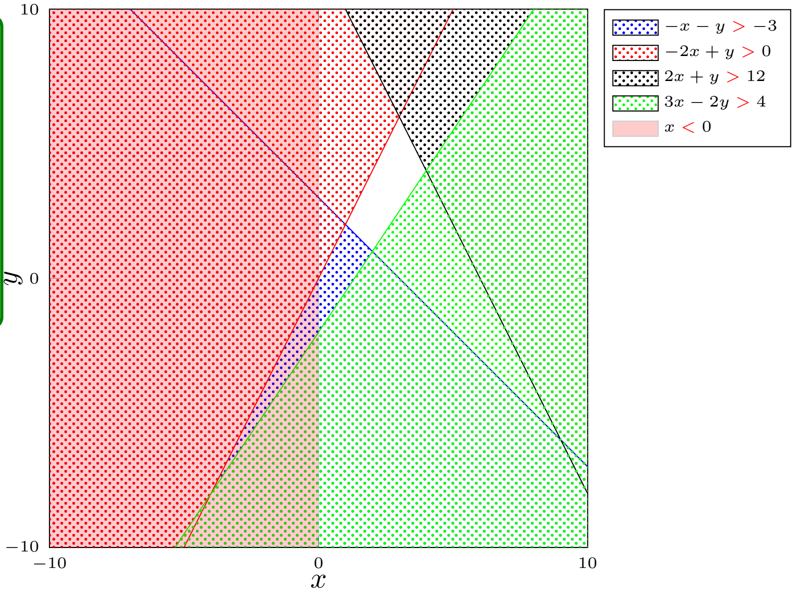

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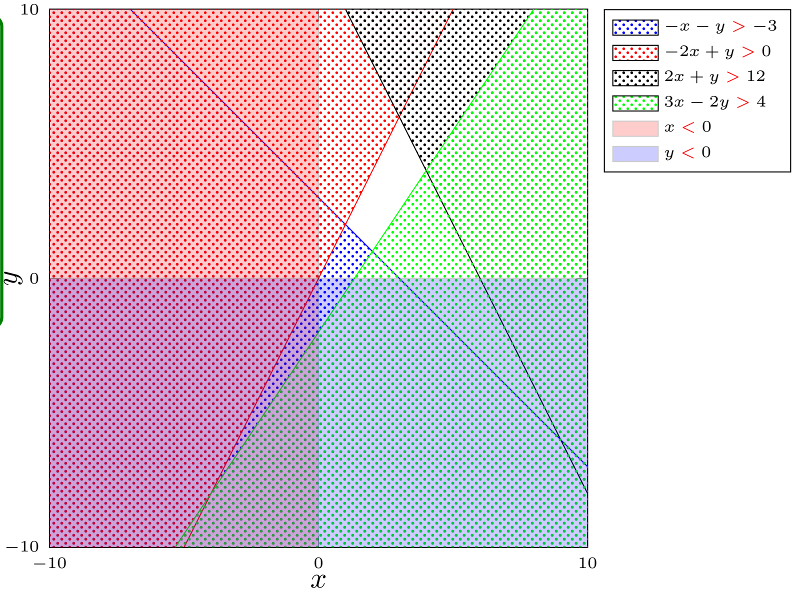

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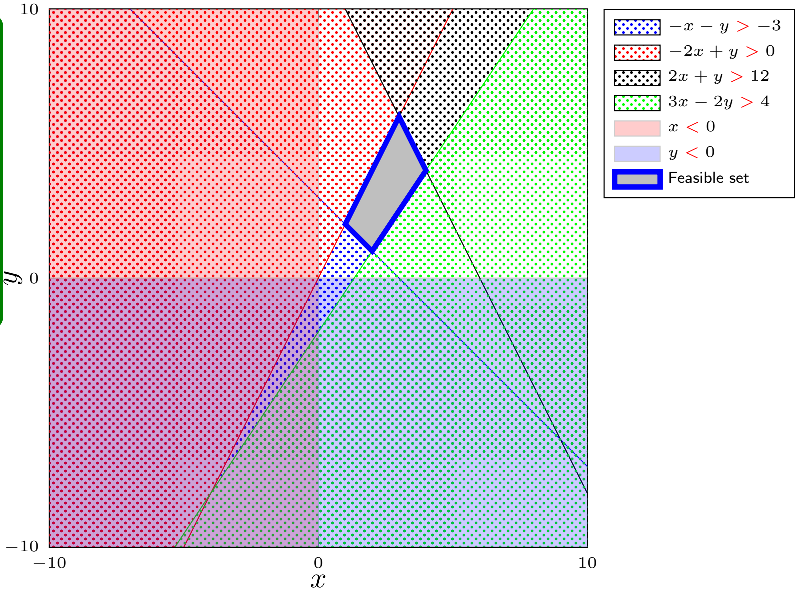

Single-objective optimization problem

Objective Function & Constraints

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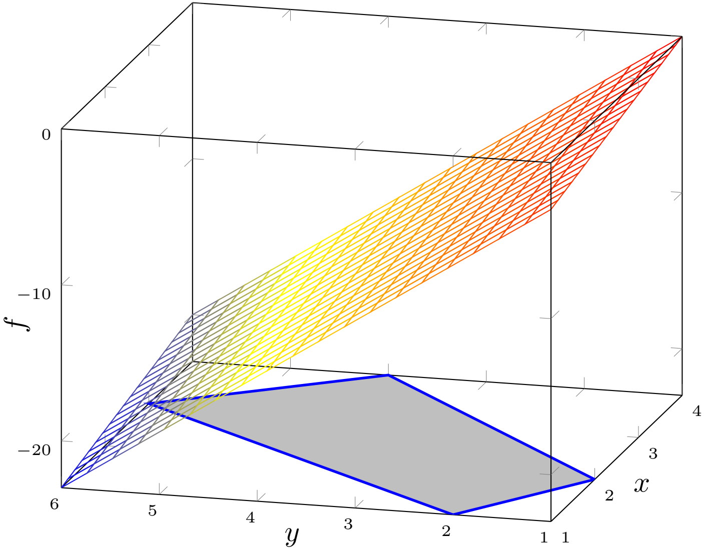

Single-objective optimization problem

Contour lines of the function f

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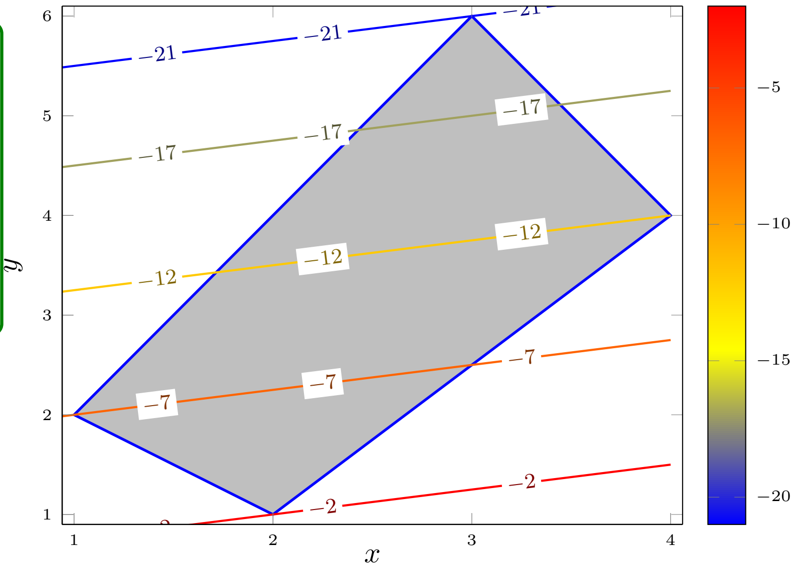

A **contour line** of a function of two variables is a curve along which the function has a constant value

Single-objective optimization problem

Gradient $-f$ & Contour lines of the function f

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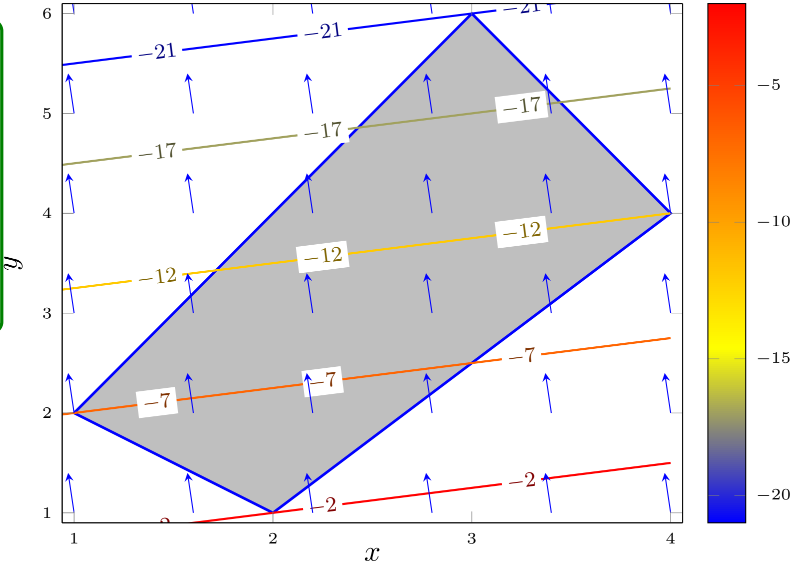

The **gradient** points in the direction of the greatest rate of increase of the function

Single-objective optimization problem

Optimal solution: $(x^*, y^*) = (3, 6)$

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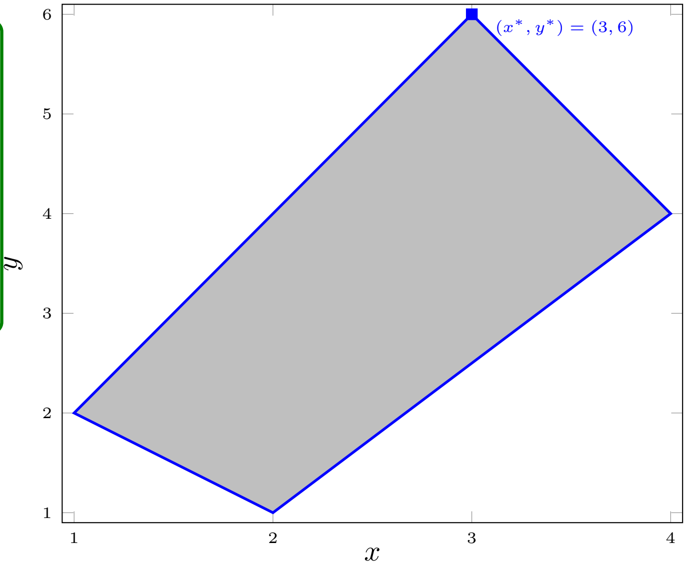

Single-objective optimization problem

Optimal solution: $(x^*, y^*) = (3, 6)$ & Optimal function value: $f^* = -21$

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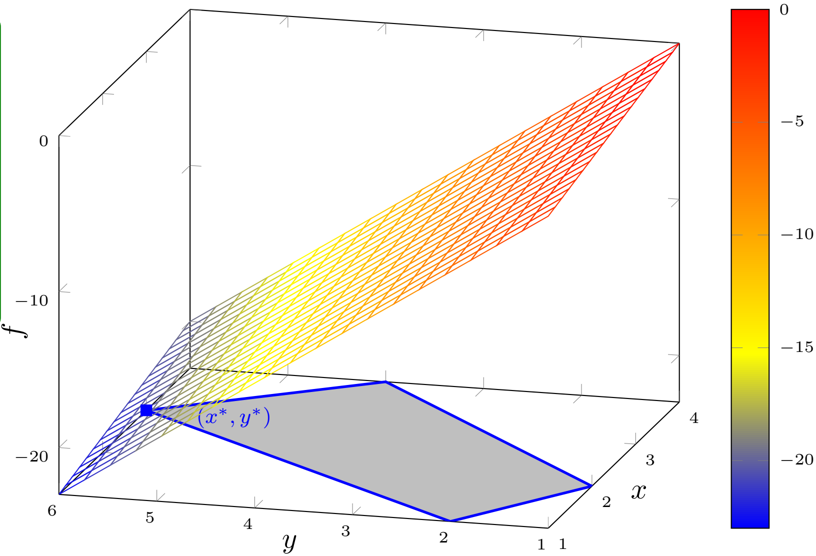

Multi-Objective Optimization problem

Two objective functions & Constraints

Example 2

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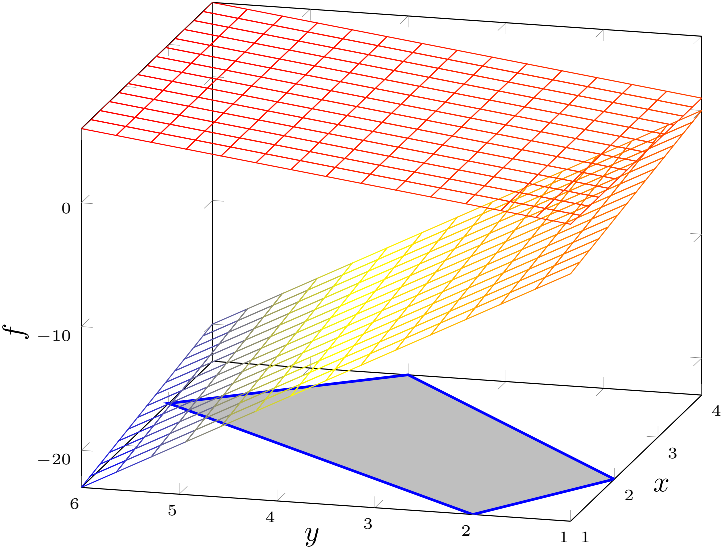

Multi-Objective Optimization Problem (MOOP)

Single vs Multi

Single-Objective Optimization Problem:

SOOP

Objective function

$$\min_{\mathbf{x}\in X} f(\mathbf{x})$$

where $X = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{g}(\mathbf{x}) \leq 0, \mathbf{h}(\mathbf{x}) = 0\}$ - **constraints**

- **Multiple objectives** are typical in real life!

Multi-objective (multicriteria, Pareto) Optimization is concerned with **simultaneous optimization** of **more than one objective function**.

- Normally objectives **conflicting** with each other:
  - Quality vs Cost
  - Efficiency vs Portability
- The scalar concept of "**optimality**" does not apply directly in MOO.

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Objective vector

$$\min_{\mathbf{x}\in X} [f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_k(\mathbf{x})]$$

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Multi-objective optimization problem

No traditional "optimality"

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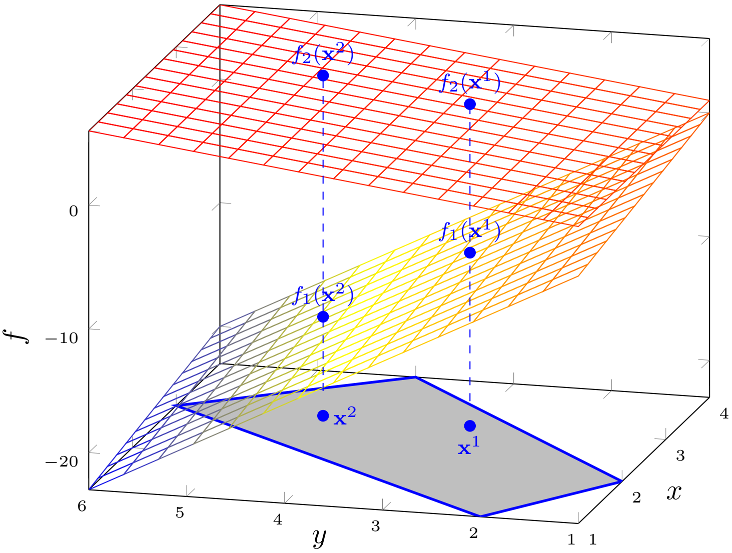

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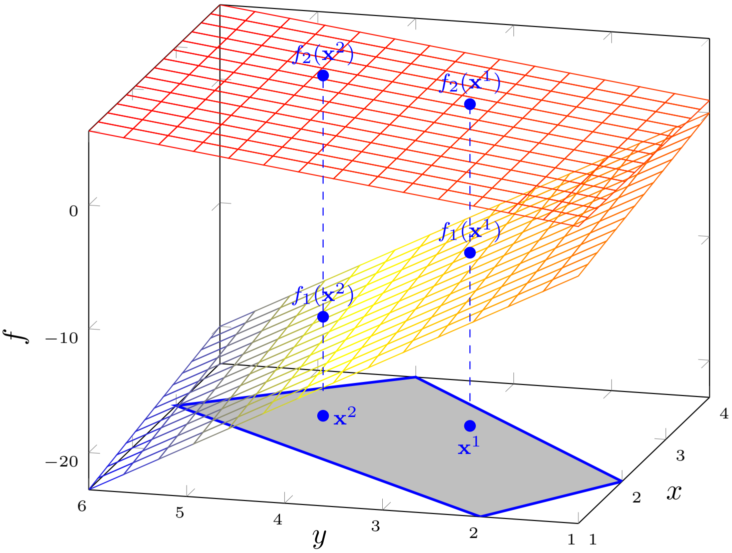


Vector $\mathbf{x}^2$ is **better** than $\mathbf{x}^1$ with respect to $f_1 : f_1(\mathbf{x}^2) < f_1(\mathbf{x}^1)$ but **worse** with respect to $f_2 : f_2(\mathbf{x}^2) > f_2(\mathbf{x}^1)$

Multiobjective Optimization

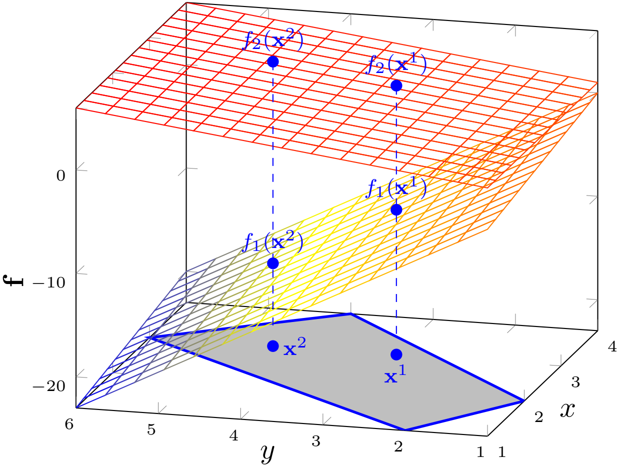
Dominated & Non-dominated vectors

Two vectors can be related to each other in two ways: either one **dominates** the other or **none of them is dominated** by the other.

Domination

A decision vector $\mathbf{x}^1 \in X$ **dominates** the vector $\mathbf{x}^2 \in X$ ($\mathbf{x^1} \succ \mathbf{x^2}$), if

1. $f_i(\mathbf{x}^1) \leq f_i(\mathbf{x}^2)$ for **all** $i \in \{1, 2, \ldots, k\}$
2. $f_j(\mathbf{x}^1) < f_j(\mathbf{x}^2)$ for **at least one** $j \in \{1, 2, \ldots, k\}$



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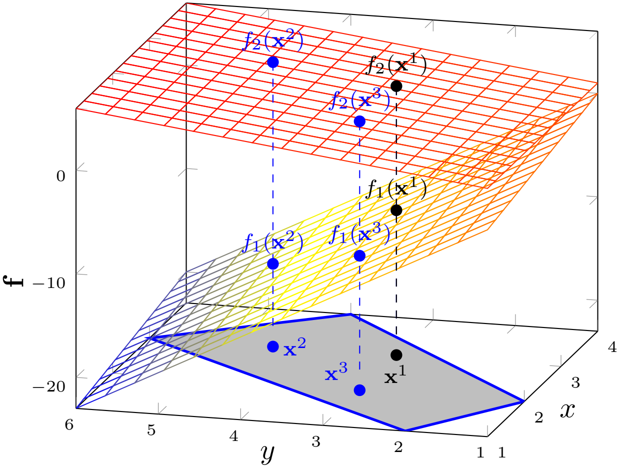
$f_2(\mathbf{x}^3) = f_2(\mathbf{x}^1)$ and $f_1(\mathbf{x}^3) < f_1(\mathbf{x}^1)$

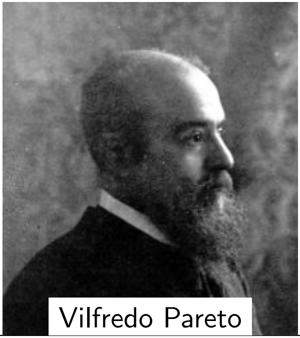


Multiobjective Optimization

Pareto optimality

A decision vector $\mathbf{x}^* \in X$ which is **non-dominated** by any other vector is called **Pareto optimal** and a set of those vectors is called **Pareto set**.

The set of corresponding objective vectors is called **Pareto front**.



Vilfredo Pareto

This definition says that $\mathbf{x}^* \in X$ is **Pareto optimal** if there exists **no feasible vector** which would **decrease** some criterion without causing a simultaneous **increase** in at least one other criterion.

The determination of these sets is the main goal of MO!

Multiobjective Optimization

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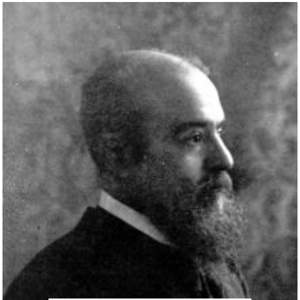
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Pareto set

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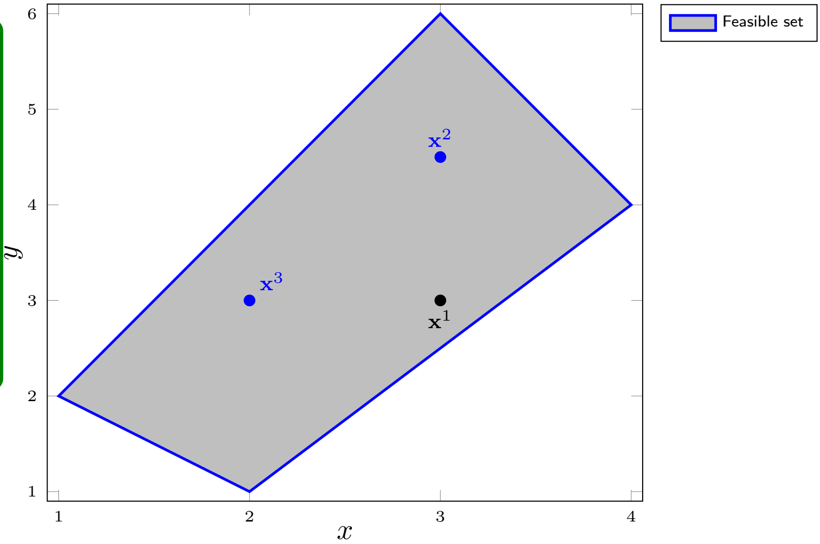

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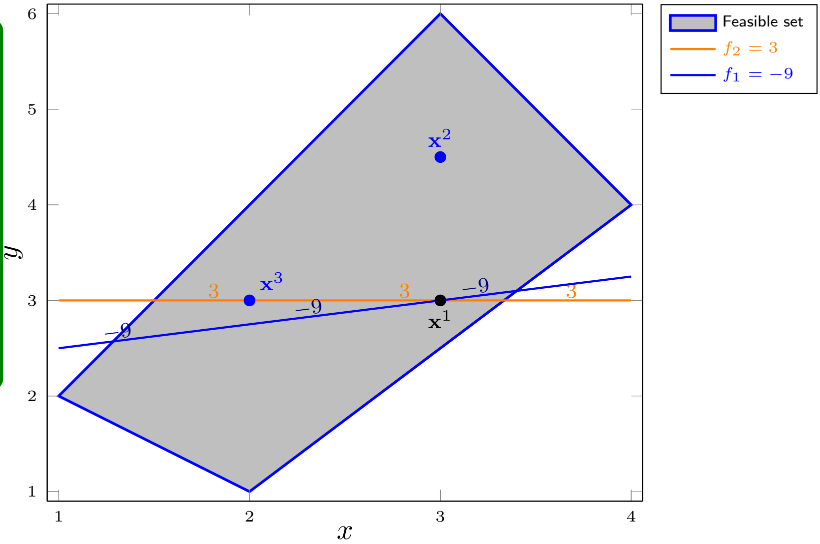

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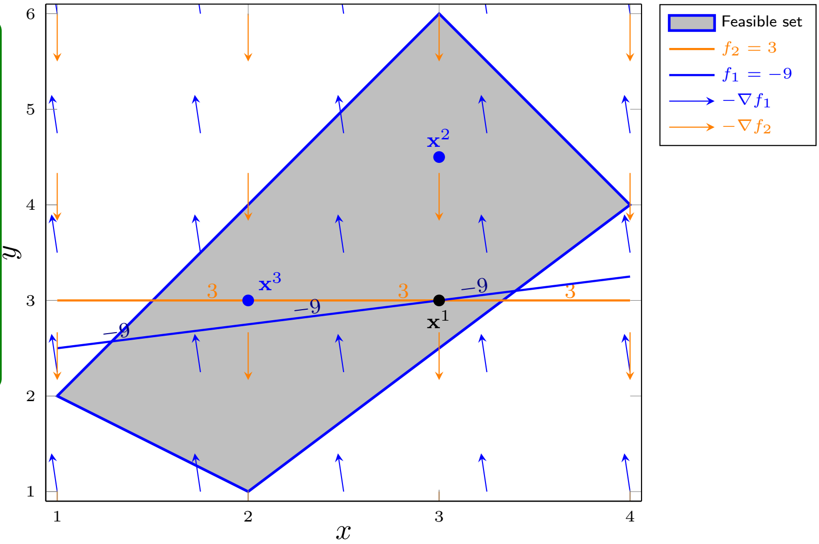

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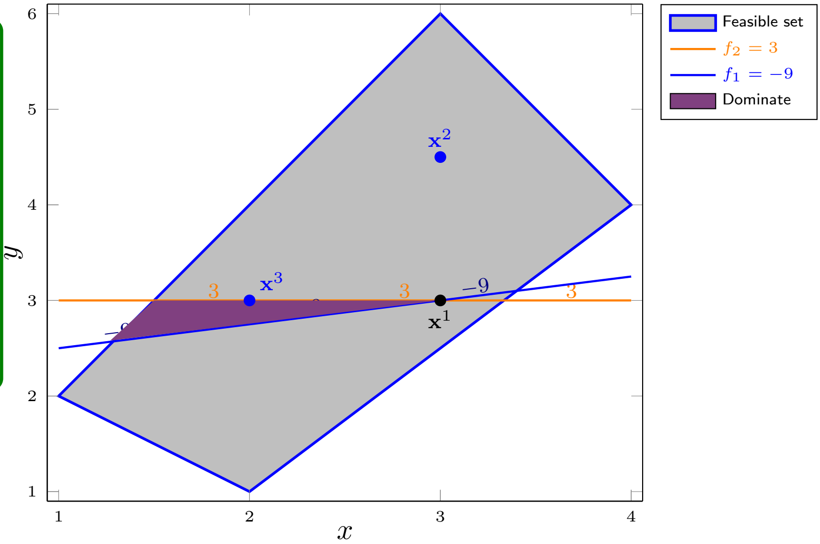
3x - 2y &\leq 4 \\
x \geq 0 \quad & \quad y \geq 0
\end{aligned}$$



Multi-objective optimization problem

Pareto set

Example 2

$$
\begin{aligned}
\min_{x,y} \ f_1(x,y) &= x - 4y \\
\min_{x,y} \ f_2(x,y) &= y \\
\text{s.t.} -x - y &\leq -3 \\
-2x + y &\leq 0 \\
2x + y &\leq 12 \\
3x - 2y &\leq 4 \\
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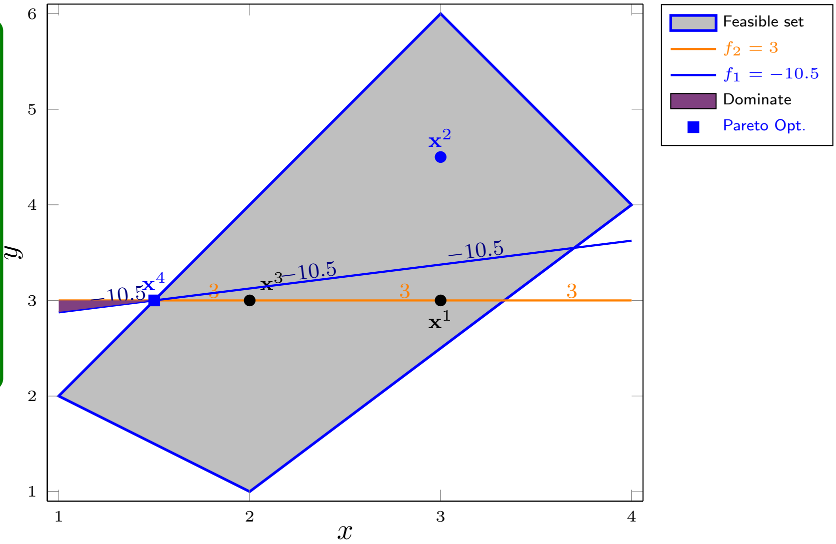
$$



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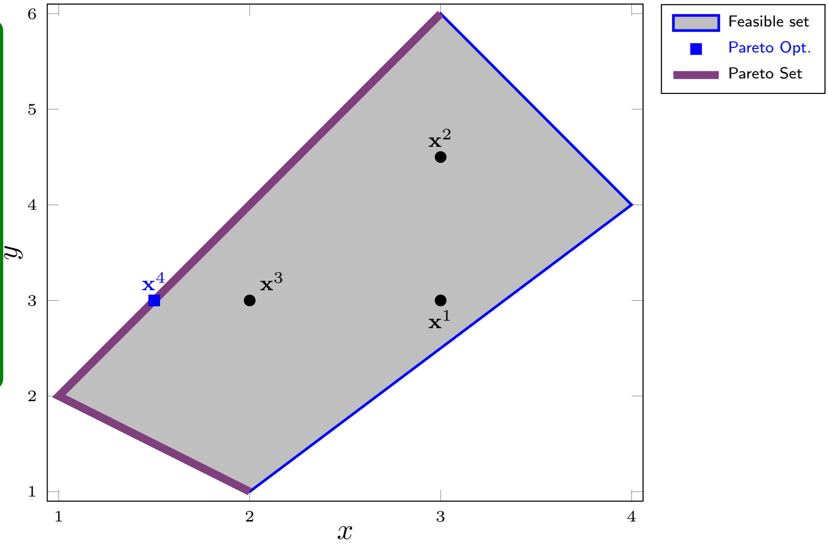
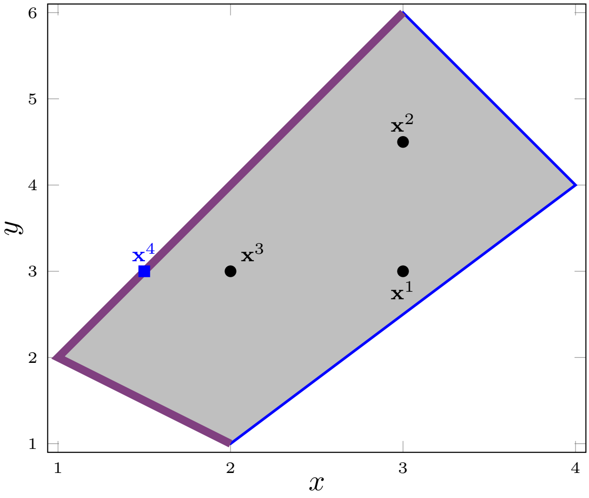


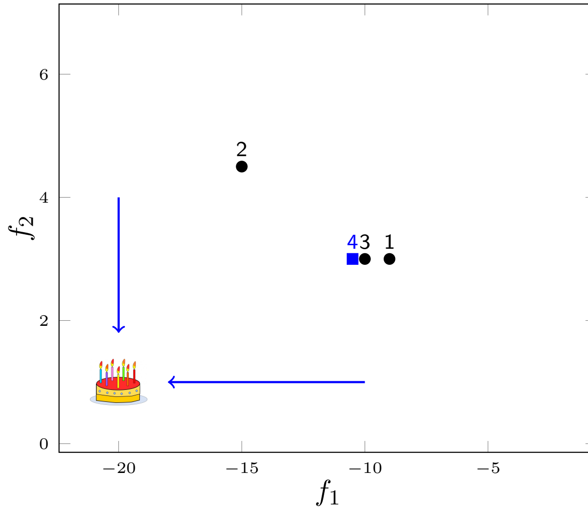
Multi-objective optimization problem

Pareto set and Pareto front

Decision space

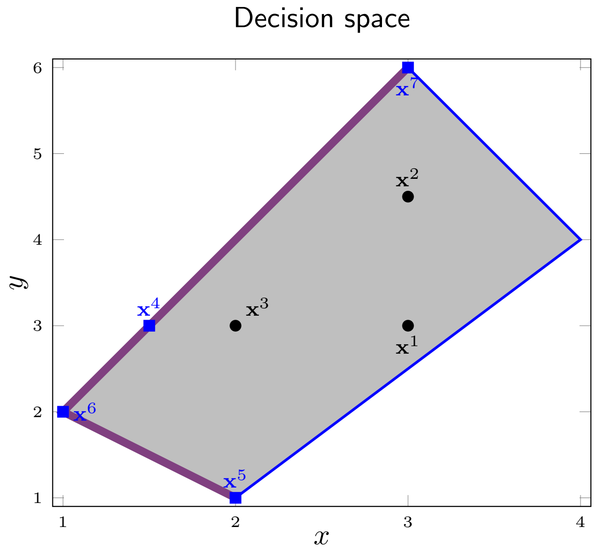


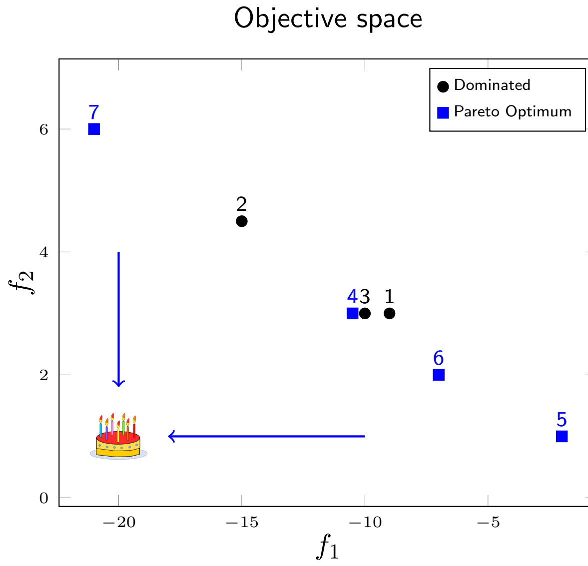
Objective space



Multi-objective optimization problem

Pareto set and Pareto front





Multi-objective optimization problem

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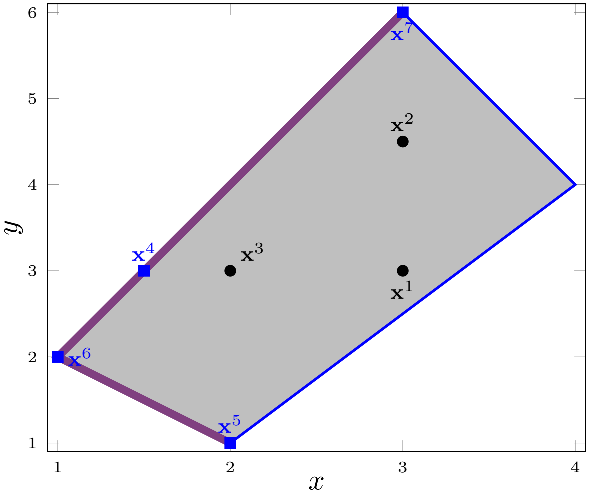

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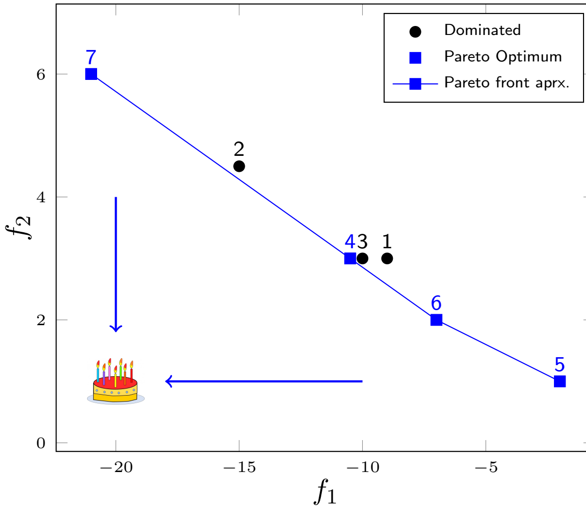

Multi-objective optimization problem

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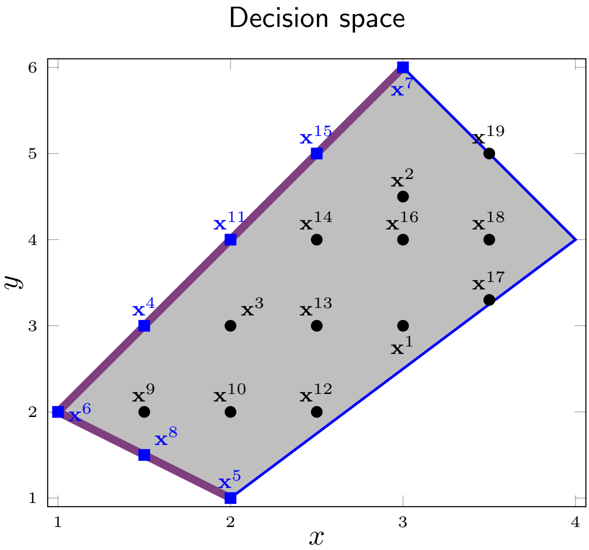

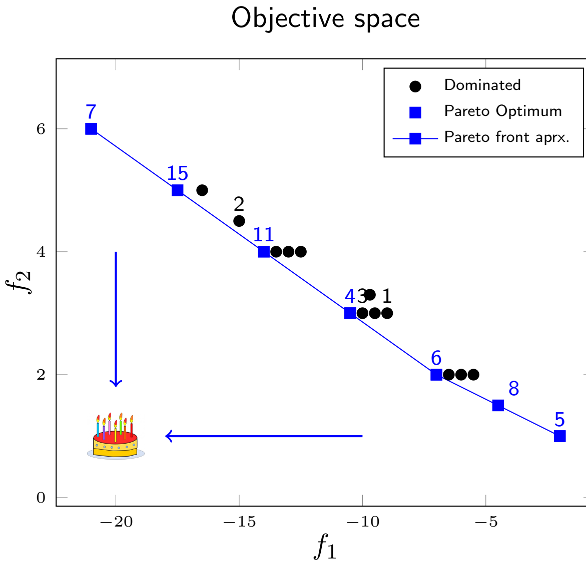

Better Pareto front approximation!

Bilevel optimization problem

The feasible region is implicitly determined by an inner optimization problem

Example 3 [Bard, 1998]

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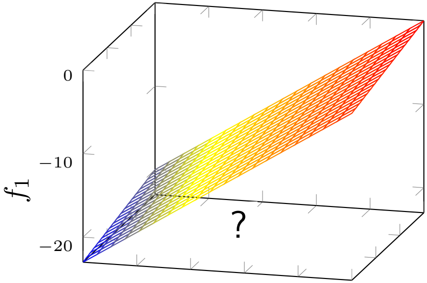

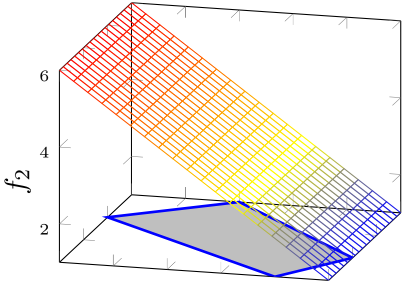

Bard, Jonathan F. (1998). Practical Bilevel Optimization, Springer US

Origins of Bilevel Programming

Mathematical programming generalization

- Consider traditional **mathematical programming** problem
- Suppose $\mathbf{y}$ to be an **optimal solution** of **nested optimization problem**
- This yields the Bilevel Programming Problem (**BPP**)[Bracken, 1973]:

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\begin{aligned}
&\min_{\mathbf{x},\mathbf{y}} F(\mathbf{x},\mathbf{y}) \\
&\text{s.t.} \quad \mathbf{G}(\mathbf{x},\mathbf{y}) \leq 0 \\
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Globally optimal solution set

Bracken J, McGill J (1973) *Mathematical programs with optimization problems in the constraints*. Operations Research 21: 37–44.

Origins of Bilevel Programming

Stackelberg game generalization

- **BPP** can be viewed as a **static** version of (non-cooperative), two-person **Stackelberg game** [Stackelberg, H., 1934].
- In **game theory** terminology, bilevel programs consist of the following rules:

1. The **leader** (the **player** that moves **first**) is **minimizing** his **objective** function F under his resource $\mathbf{G}$. This **sub-problem** is called the **leader's** (**outer** or **upper-level**) **problem**.
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3. The **leader's** decision variables are $\mathbf{x}$ while the **follower's** are $\mathbf{y}$.

Stackelberg H (1934) The Theory of Market Economy. Oxford University Press, UK

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"Bilevel Game"

4. The **game** between the leader and the follower consists of these steps:

   4.1 The **leader** chooses a value of $\hat{\mathbf{x}} \in X$.

   4.2 The **follower** formulates a best response based on $\hat{\mathbf{x}}$ and generates the set $\mathcal{Y}(\hat{\mathbf{x}})$ (each $\mathbf{y} \in \mathcal{Y}(\hat{\mathbf{x}})$ is an **optimal response** for the **follower** of given $\hat{\mathbf{x}}$ from the **leader**).

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BPP formulation

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$$\text{s.t. } \mathbf{G}(\mathbf{x}, \mathbf{y}) \le 0$$

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Globally optimal solution set

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$\mathcal{Y}(\hat{\mathbf{x}}) \longleftarrow$

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Globally optimal solution set

$$\mathbf{x} \in X \subset \mathbb{R}^n, \mathbf{y} \in \mathcal{Y}(\mathbf{x})$$

Bilevel optimization problem

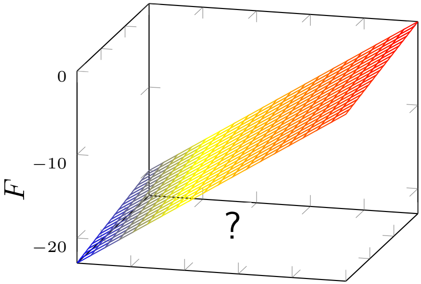
Nested optimization problem within the constraints of another optimization problem

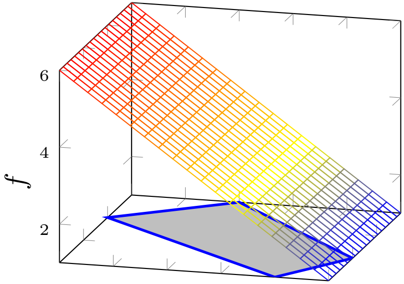
Example 3 [Bard, 1998]

$$\min_{x,y} F(x,y) = x - 4y$$

$$\text{s.t.} \quad \min_{y} f(x,y) = y$$

$$\begin{aligned} \text{s.t.} \quad -x - y &\le -3 \\ -2x + y &\le 0 \\ 2x + y &\le 12 \\ 3x - 2y &\le 4 \\ x \ge 0, y &\ge 0 \end{aligned}$$





Bilevel optimization problem

Constraint region

$$S = \{(\mathbf{x},\mathbf{y}) : \mathbf{x} \in X, \mathbf{y} \in Y, \mathbf{G}(\mathbf{x},\mathbf{y}) \le 0, \mathbf{g}(\mathbf{x},\mathbf{y}) \le 0\}.$$

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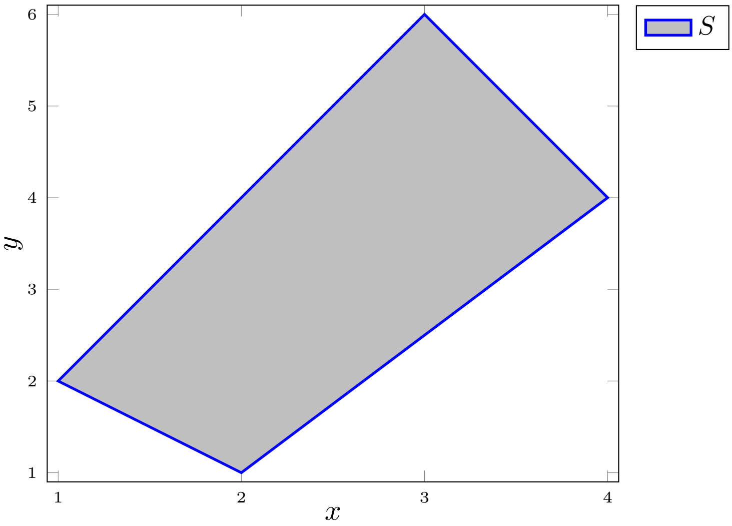


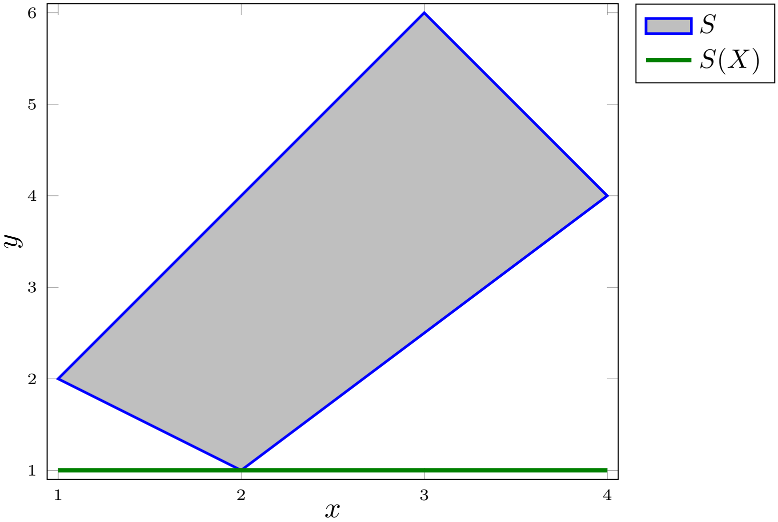
Bilevel optimization problem

Projection of S onto the **leader's** decision space

$$S(X) = \{\mathbf{x} \in X : \exists \mathbf{y} \in Y, \mathbf{G}(\mathbf{x},\mathbf{y}) \leq 0, \mathbf{g}(\mathbf{x},\mathbf{y}) \leq 0\}.$$

Example 3 [Bard, 1998]

$$\begin{aligned} \min_{x,y}\ & F(x,y) = x - 4y \\ \text{s.t.}\ & \min_{y} f(x,y) = y \\ & \quad \text{s.t.}\ -x - y \leq -3 \\ & \quad\quad -2x + y \leq 0 \\ & \quad\quad 2x + y \leq 12 \\ & \quad\quad 3x - 2y \leq 4 \\ & \quad\quad x \geq 0, y \geq 0 \end{aligned}$$



Bilevel optimization problem

Follower's feasible set for each fixed $\hat{x} \in S(X)$

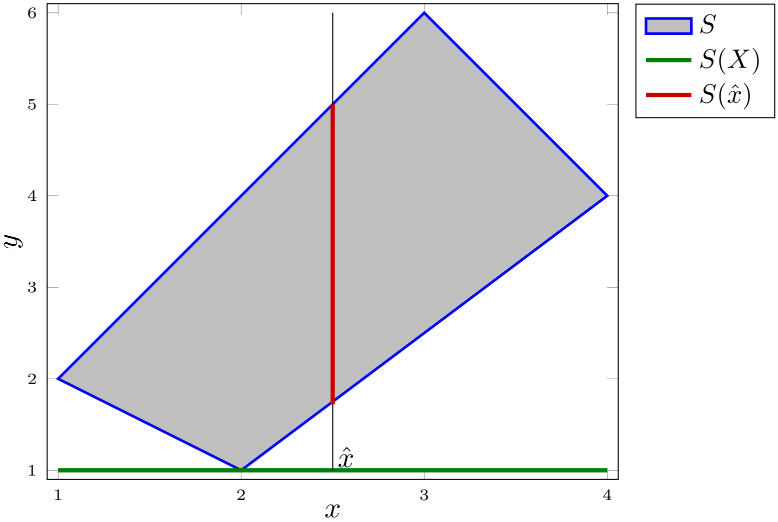
$$S(\hat{\mathbf{x}}) = \{\mathbf{y} \in Y : \mathbf{g}(\hat{\mathbf{x}}, \mathbf{y}) \leq 0\}.$$

Example 3 [Bard, 1998]

$$\min_{x,y} F(x,y) = x - 4y$$

$$\text{s.t.} \quad \min_{y} f(x,y) = y$$

$$\begin{aligned} \text{s.t.} \quad -\hat{x} - y &\leq -3 \\ -2\hat{x} + y &\leq 0 \\ 2\hat{x} + y &\leq 12 \\ 3\hat{x} - 2y &\leq 4 \\ x = \hat{x}, y &\geq 0 \end{aligned}$$



Bilevel optimization problem

Follower's rational reaction set for each $\hat{x}$

$$\mathcal{Y}(\hat{\mathbf{x}}) = \{\mathbf{y} \in Y : \min\{f(\hat{\mathbf{x}}, \mathbf{y}) : \mathbf{y} \in S(\hat{\mathbf{x}})\}\}.$$

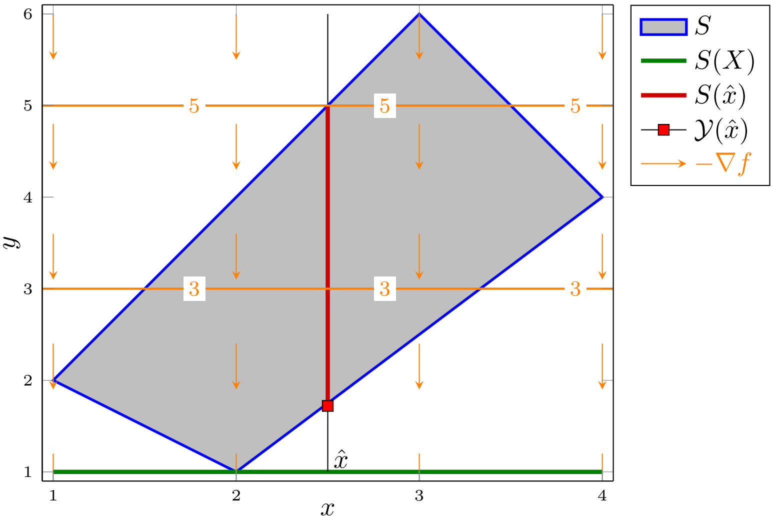
Defines the follower's response

Example 3 [Bard, 1998]

$$\min_{x,y} F(x,y) = x - 4y$$

$$\text{s.t.} \quad \min_{y} f(\hat{x}, y) = y$$

$$\begin{aligned} \text{s.t.} \quad -\hat{x} - y &\leq -3 \\ -2\hat{x} + y &\leq 0 \\ 2\hat{x} + y &\leq 12 \\ 3\hat{x} - 2y &\leq 4 \\ x = \hat{x}, y &\geq 0 \end{aligned}$$



Bilevel optimization problem

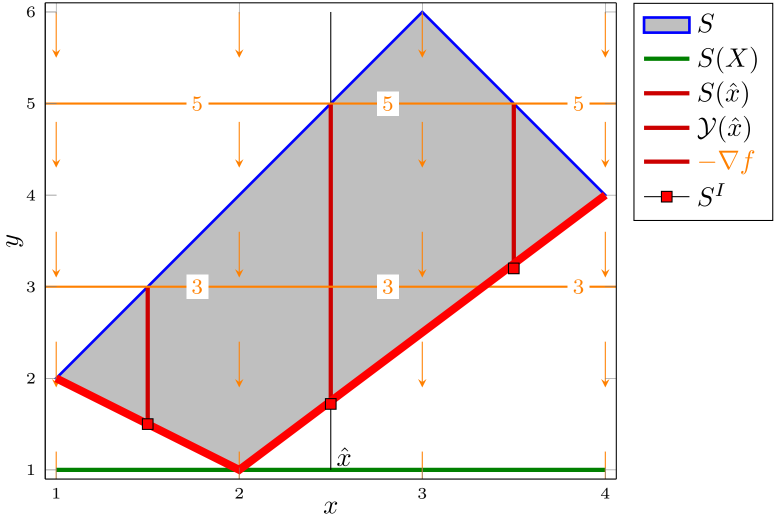
The feasible set of bilevel problem (Inducible region)

$$S^I = \{(\mathbf{x}, \mathbf{y}) : (\mathbf{x}, \mathbf{y}) \in S, \mathbf{y} \in \mathcal{Y}(\mathbf{x})\}.$$

The **induced region** is usually **nonconvex** and, in the presence of **upper level constraints**, can be **disconnected** or even **empty**.

Example 3 [Bard, 1998]

$$\begin{aligned} \min_{x,y}\ & F(x,y) = x - 4y \\ \text{s.t.}\ & \min_{y} f(x,y) = y \\ & \text{s.t.}\ -x - y \le -3 \\ & \quad -2x + y \le 0 \\ & \quad 2x + y \le 12 \\ & \quad 3x - 2y \le 4 \\ & \quad x \ge 0, y \ge 0 \end{aligned}$$



Bilevel optimization problem

Compact BPP formulation

Given these definitions the BPP can be **reformulated** as:

$$\min \{F(\mathbf{x}, \mathbf{y}) : (\mathbf{x}, \mathbf{y}) \in S^I\}.$$

Example 3 [Bard, 1998]

$$\begin{aligned}
\min_{x,y}\ & F(x,y) = x - 4y \\
\text{s.t.}\ & \min_{y} f(x,y) = y \\
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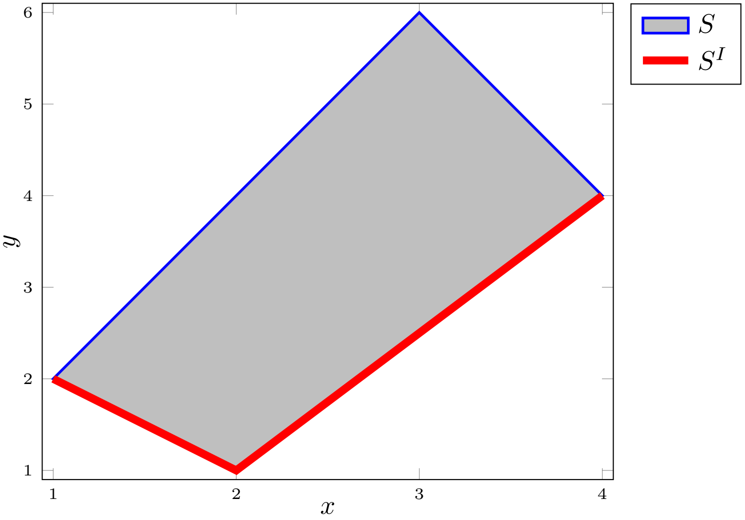
& \qquad -2x + y \leq 0 \\
& \qquad 2x + y \leq 12 \\
& \qquad 3x - 2y \leq 4 \\
& \qquad x \geq 0, y \geq 0
\end{aligned}$$



Bilevel optimization problem

Gradient $(-\nabla F) \uparrow$ & Contour lines: $F=-2; F=-7; F=-12; F=-21$

Example 3 [Bard, 1998]

$$\min_{x,y} F(x,y) = x - 4y$$
$$\text{s.t.} \quad \min_{y} f(x,y) = y$$
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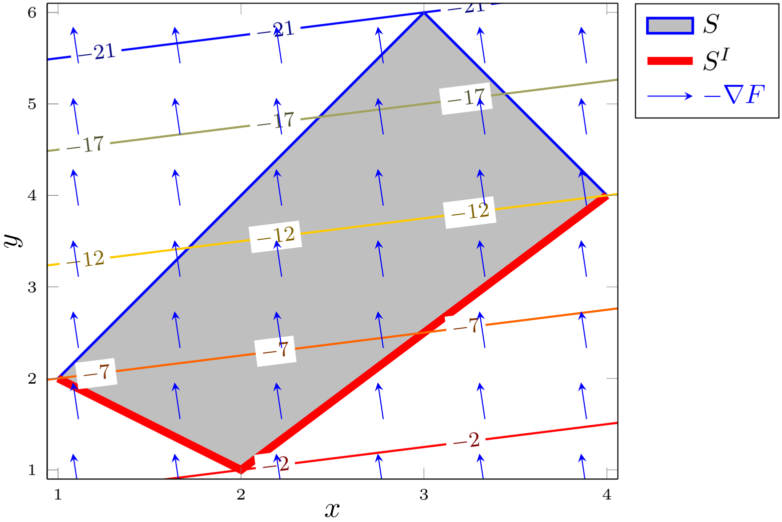

Bilevel optimization problem

Optimal solution: $(x^*, y^*) = (4, 4)$

Example 3 [Bard, 1998]

$$
\begin{aligned}
\min_{x,y} \; & F(x,y) = x - 4y \\
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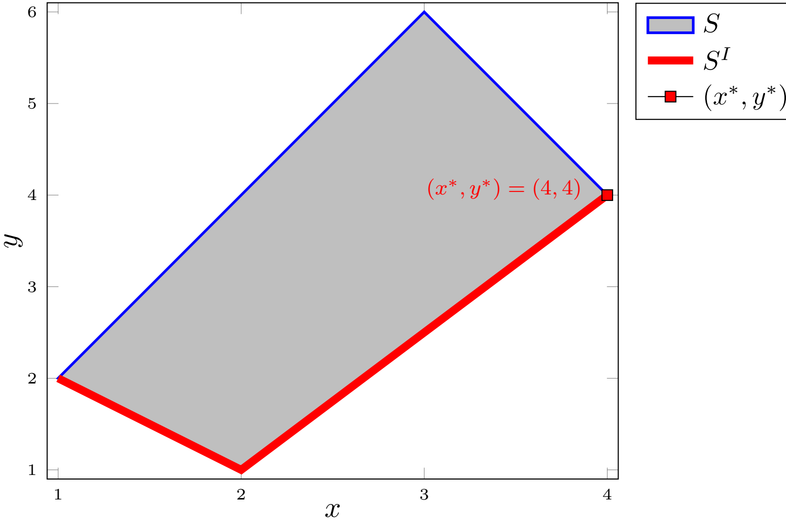
$$



Bilevel optimization problem

Optimal solution: $(x^*, y^*) = (4, 4)$ & Optimal function values: $F^* = -12$ and $f^* = 4$

Example 3 [Bard, 1998]

$$\min_{x,y} F(x,y) = x - 4y$$
$$\text{s.t.} \quad \min_{y} f(x,y) = y$$
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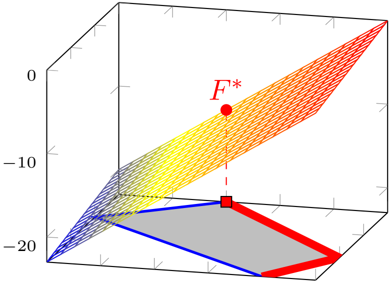
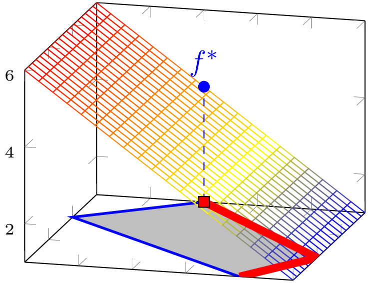
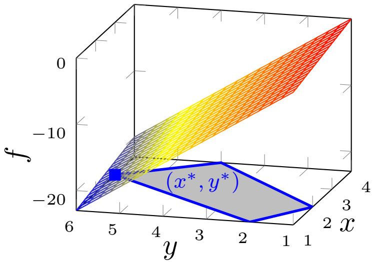
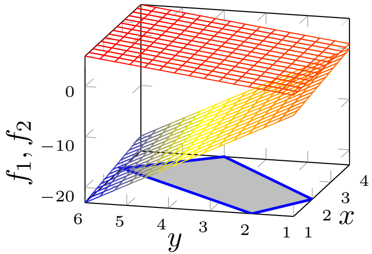
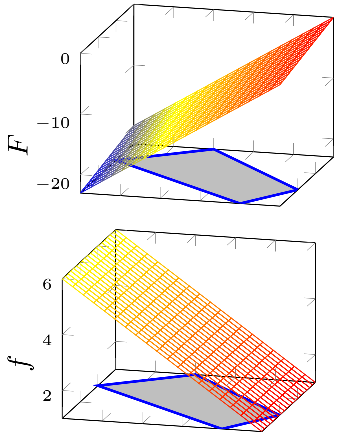




Summary: Single level vs Multi-objective vs Bilevel Optimization

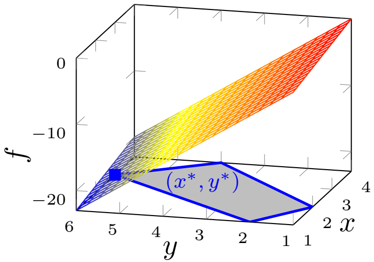


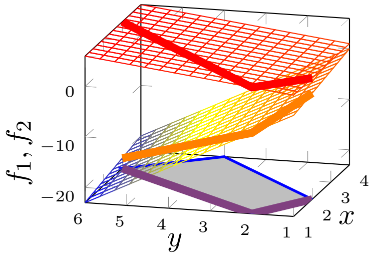


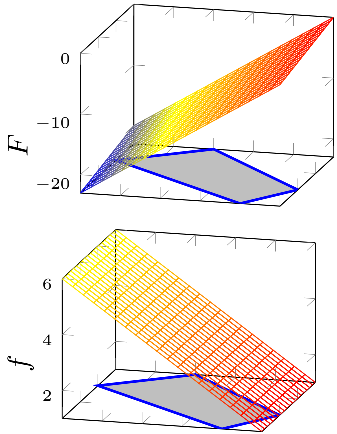


- **Single-level problems** usually have a **single** optimal **solution**.
- **Multi-objective** problems **usually** have **multiple** optimal **solutions**.
- **Bilevel problems** usually also have a **single** optimal **solution**.
- A **bilevel solution** is **not** necessarily a **Pareto-optimal** solution.
- It is **not possible** to directly **use algorithms** for multi-objective optimization for bilevel problems.

Summary: Single level vs Multi-objective vs Bilevel Optimization







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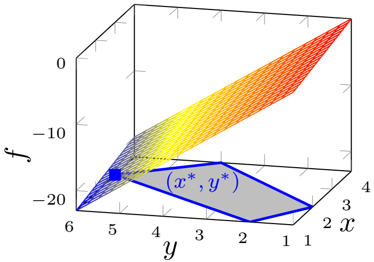

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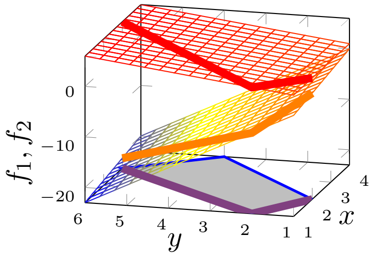

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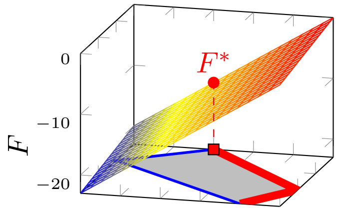

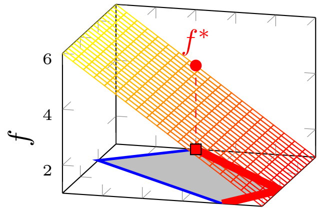

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Thank You For Your Attention!

Questions ?

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