A New Classification of One-Dimensional Cellular Automata Using Grossone

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The One-Dimensional Plan

- Re-Introduce (Recall) The Infinite Unit Axiom and Grossone
- Define and Discuss One-Dimensional Cellular Automata
- Apply Grossone to Define a new Metric on the Space of One-Dimensional Cellular Automata
- Apply this new Metric and Grossone to Develop a Classification of One-Dimensional Cellular Automata

Infinite Unit Axiom

Enhances the concept of the unit from finite to infinite.

The number of elements in the set N, of natural numbers is equal to the infinite unit denoted by ①

We will give this a name: “Grossone”

Introduced in the early part of the 21st century by Yaroslav Sergeyev

The Properties

Infinity: For any finite natural number n it follows $n < ①$

Identity: The following relations link to identity elements 0 and 1:

1.) $0 \cdot ① = 0 = ① \cdot 0$

2.) $① - ① = 0$

3.) $① / ① = 1$

4.) $①^0 = 1$

5.) $1^{①} = 1$

Divisibility. For any finite natural number n sets

$N_{k,n} = \{k, k+n, k+2n, k+3n, \ldots\}, \quad 1 \leq k \leq n,$

being the n-th parts of the set, **N**, of natural numbers have the same number of elements indicated by the numeral ①/n.

Divisibility

These numbers are defined as the number of elements in the *nth* part of the set N

$\textcircled{1}$: {1, 2, 3, 4, 5, 6, 7, ...}

$\frac{\textcircled{1}}{2}$: {1, _ 3, _ 5, _ 7, ...}

$\frac{\textcircled{1}}{2}$: {_ 2, _ 4, _ 6, _ ....}

$\frac{\textcircled{1}}{3}$: { 1, _ _, 4, _ _, 7, ...}

$\frac{\textcircled{1}}{3}$: {_ 2, _ _, 5, _ _, ...}

$\frac{\textcircled{1}}{3}$: {_ _ 3, _ _ 6, _ ....}

Note: The '_' means the natural numbers that would have occupied these places have been excluded from the set N of naturals

Some Notes

- The new approach does not contradict Cantor, and it can be viewed as an evolution of his ideas regarding the existence of different infinite numbers in a more applied and precise way.
- The Infinite Unit Axiom introduces a new infinite number, Grossone, and the properties that distinguish it from other numbers.
- ① - 3, 2 ①10, 5①^{2} (-2)①4.2①$^{-2}$ are all numbers in this new positional infinite base number system.
  - Note that ①$^{-2}$ is an infinitesimal.
  - For example, 2①10 = 2① + 10

Infinitesimals

- Infinite numbers with parts of the type ①$^{-i}$, with $i > 0$ are called *infinitesimals*
- Infinitesimals will play an important role in defining a metric and analyzing the forward evolution of cellular automata.

Why Not Use The Hyperreals/Infinitesimals?

- Grossone is defined as a number (an infinite number)
- Computation Power
  - Grossone, with the defined properties, provides computational power (we have representations of infinite numbers).

Some Important Sets

N, the set of natural numbers, is:

$\{1, 2, 3, 4, ..., ① -2, ① -1, ①\}$

The set Z, of integers, is:

$\{-①, -①+1, ..., -2, -1, 0, 1, 2, 3, ..., ① -2, ① -1, ①\}$

For further information, please see:
http://www.grossone.com/arithmetic.html

Extended Sets

By adding the *Infinite Unit Axiom* to the axioms of natural numbers

The set $\hat{N}$ of extended natural numbers is formed:

$\{1, 2, ..., ① -2, ① - 1, ①, ① +1, ..., ①^2 - 1, ①^2, ①^2 +1, ...\}$

Sure, we have:

$1 < 2 < ... < ① - 1 < ① < ① + 1 < ... < ①^2 - 1, < ①^2 < ...$

The set $\hat{Z}$ of *extended integers* is constructed from Z

This is accomplished the same way as the natural numbers but with additive inverse elements. For example, 2① has as its additive inverse -2①

Number of elements in Some Important Sets

In N, the set of natural numbers, there are ① elements

In the set E, set of evens, there are $\frac{①}{2}$ elements

The same is true for the set of odd numbers.

The set Z, of integers, has $2① + 1$ elements

It can be shown that $|Q| = 2①^2 + 1$

The set Z × Z has $(2① + 1) * (2① + 1) = (2① + 1)^2$ elements or $4①^2 + 4① + 1$ elements

For further information, please see:
http://www.grossone.com/arithmetic.html

Cellular Automata

- Discrete Systems
- Known for their strong modeling properties
- Can exhibit self-organizational behavior
- Are capable of universal computation

Why All the Hype with Cellular Automata?

- Developed by Von Neumann and Ulam to model physical and biological systems
- Actually they developed the concept for parallel computation whereby a system of local machines update themselves in parallel according to a local rule.
- Discrete (dynamical) systems to model continuous systems

Some Applications

- Universal Computation (Turing Machines)
- Parallel Computation
- Lattice Gas Theory
- Forest Fire Models
- Lava Flow Models
- Cellular and Bacteria Growth Models
- Traffic Flow Models

Definitions and Introduction (One-Dimensional)

- An alphabet S of size greater than 1
  - For example, S = {0,1} (the binary alphabet)
- Use the one-dimensional integer lattice Z and let $X = S^Z$
  - The space of all maps $x: Z \to S$
    - Or the space of all bi-infinite sequences of elements of S

The setting

(la regolazione)

One-Dimensional Cellular Automata Maps

- One-dimensional cellular automata are induced by arbitrary maps (local rule): $F: S^{(2r+1)} \longrightarrow S$
  - $r \in N \cup \{0\}$ is called the range of the map.
- The automaton map f induced by F is defined by f(x) = y with $y(i)=F(x[i-r],\ldots,x[i],\ldots,x[i+r])$

One-Dimensional Neighborhoods

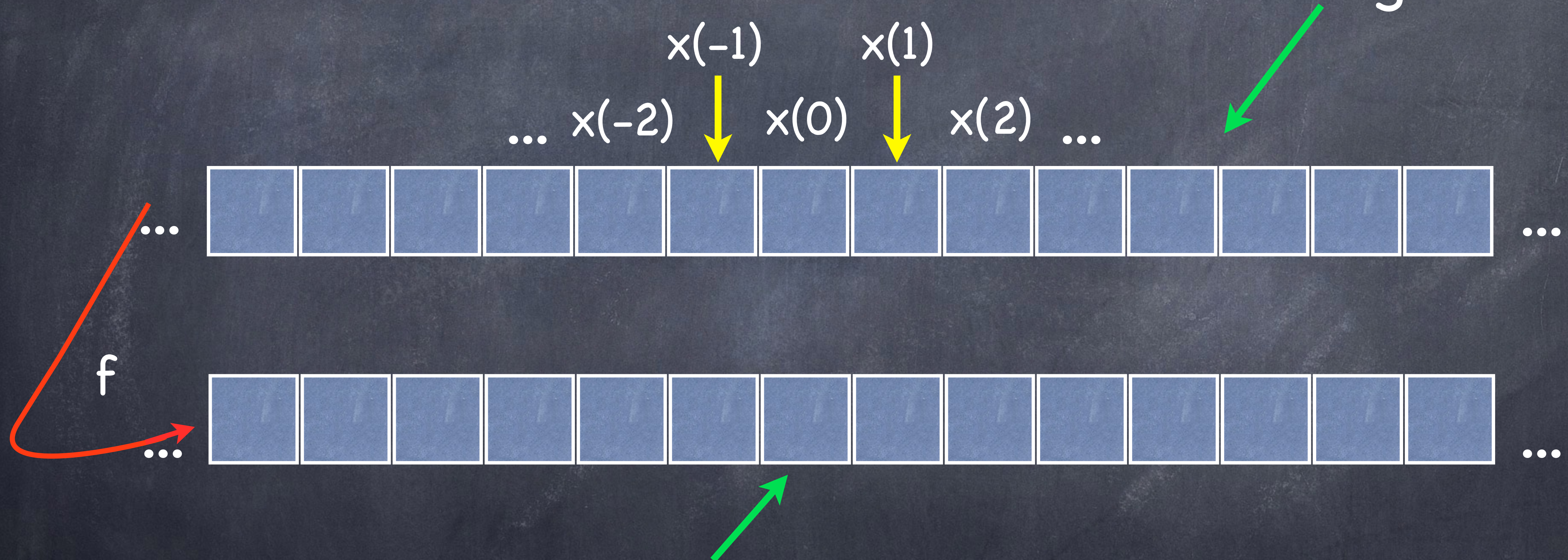
A neighborhood of range r = 1

A neighborhood of range r = 2

A Configuration

Defined on the integers, Z:

The initial configuration



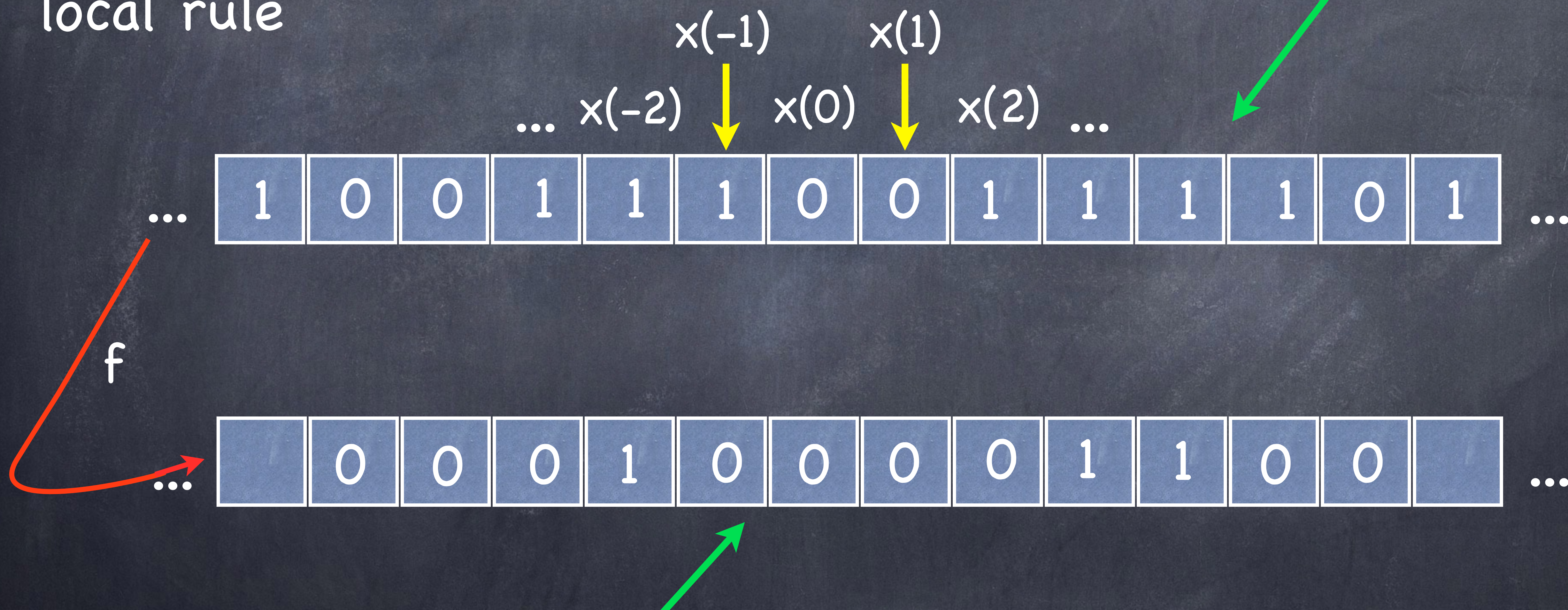
The first iteration (application) of the automaton local rule F, in parallel

An Example

$$F(a,b,c) = \begin{cases} 1 & \text{if } a=b=c=1 \\ 0 & \text{otherwise} \end{cases}$$

local rule

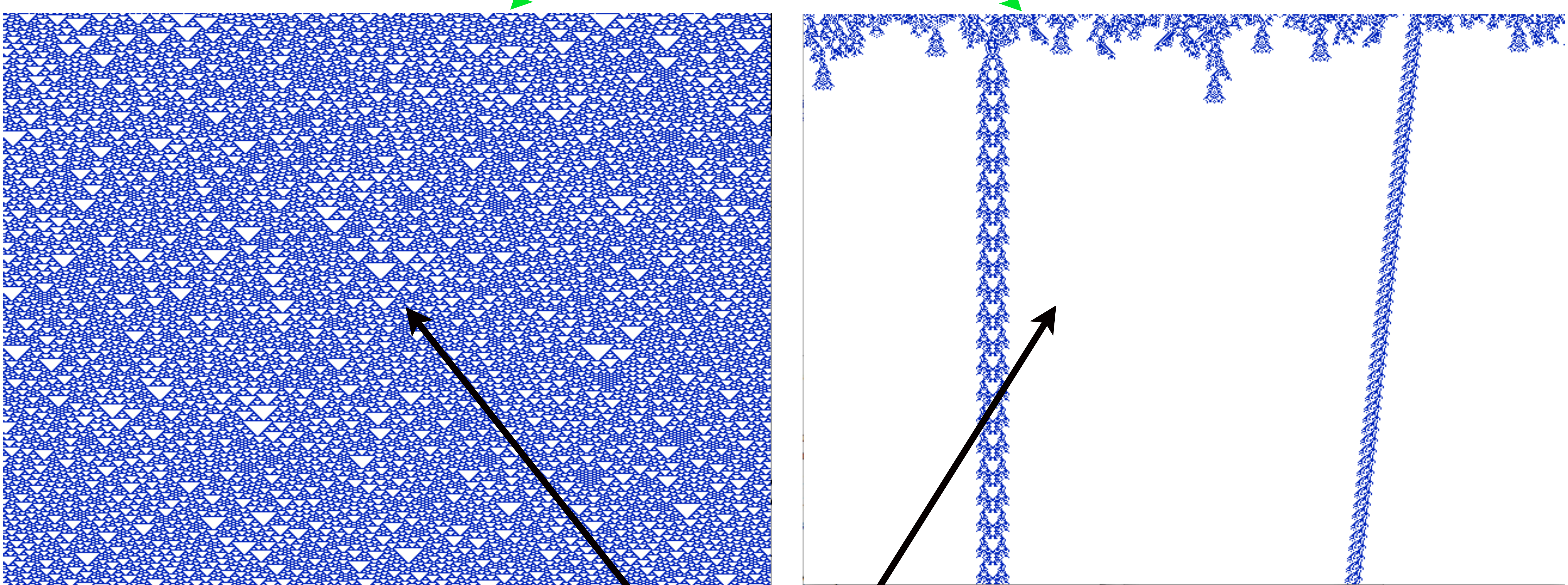
The initial configuration



The first iteration (application) of the automaton function f

Some Interesting Cellular Automata Rules and their Forward Evolution

Initial Random Configurations



Forward Iterations (Evolution)

A Dynamical Process

- Cellular Automata produce a dynamic process (Discrete) -an evolutionary (iterative) process
  - Hence it is interesting to study the long term effects of these processes.
  - 1980s : Stephen Wolfram studied the forward dynamics of **<u>one</u>-dimensional** cellular automata
    - Noticed that different configurations behave differently under different automata rules.
    - However, if the configuration is chosen at random, the probability is high that the CA will fall into one of 4 classes

A Dynamical Process (formal definition)

The set N of natural numbers = {1, 2, 3,...., ① - 1, ①}

The set N_0 = {0,1, 2, 3,...., ① - 1, ①}

The i^{th} iterate of a function (automaton function) is the sequence:

$$f^{i}(x) = f \circ f \circ f \circ f \circ f \ldots \circ f(x) \quad \text{where} \quad 0 \leq i \leq ①$$

<u>Theorem</u>: The number of elements of any infinite sequence is less than or equal to ① [1]

[1] Sergeyev, A new Applied Approach for executing computations with infinite and infinitesimal quantities, *Informatica* 19 (4) (2008) 567-596

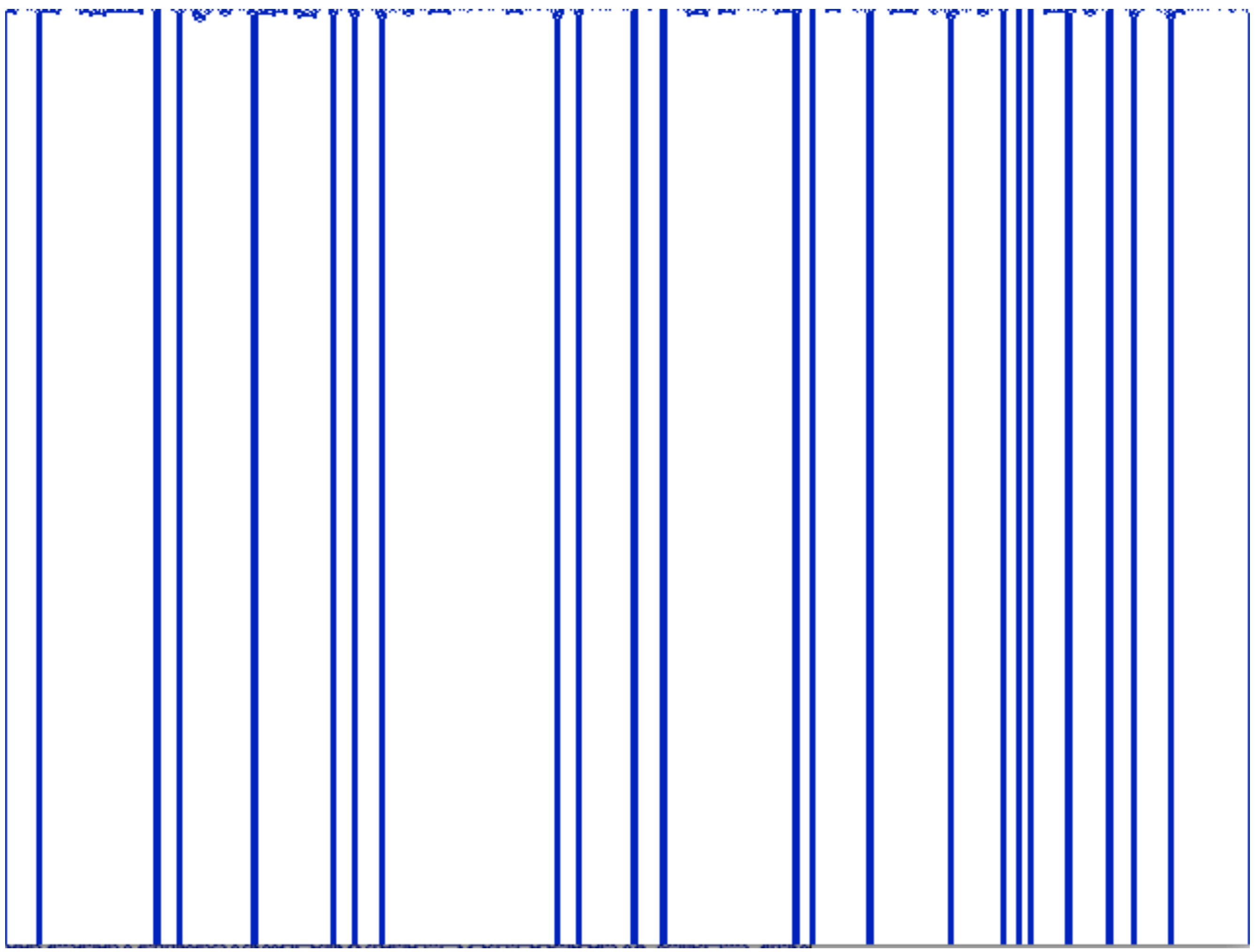
Wolfram Classes

Partitioned one-dimensional CA into 4 classes based on their observed dynamical behavior:

- Class 1: Evolution tends to a spatially homogeneous state
- Class 2: Evolution yields simple stable or periodic structures
- Class 3: Exhibits chaotic aperiodic behavior
- Class 4: Yields complex localized structures, some propagating

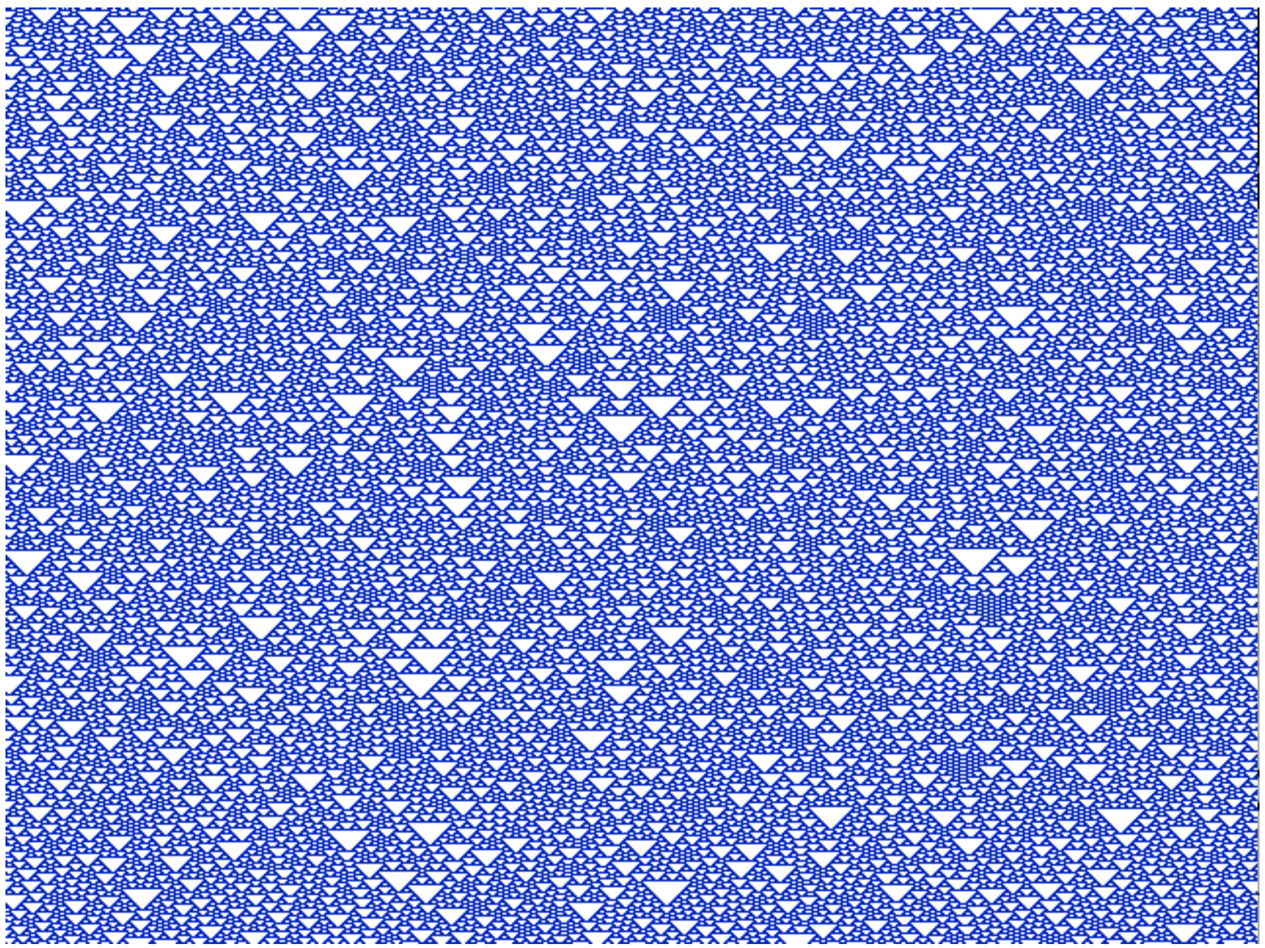
Some Cellular Automata Rules

Wolfram Class 1
Rule 36

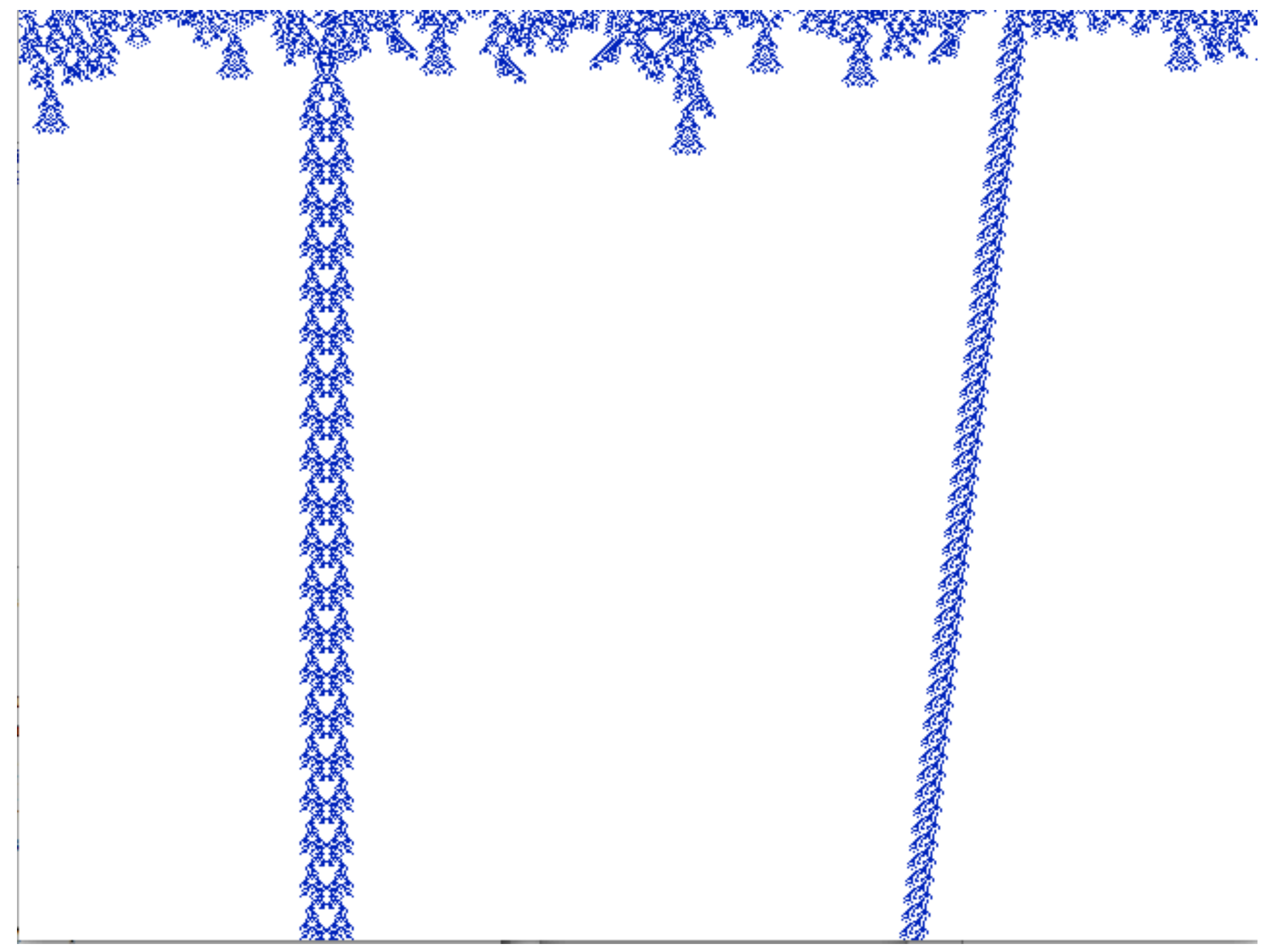


Wolfram Class 2
Rule 24

More Cellular Automata Rules



Wolfram Class 3
Rule 12



Wolfram Class 4
Rule 20

A More Rigorous Classification Scheme of One-Dimensional CA

- Developed in the mid 1980s by Robert Gilman
  - Measure theoretic classification based on the probability of finding another sequence (configuration) that stays arbitrarily close to a given initial configuration.
    - Use the metric $d(x,y) = 2^{-n}$, where $n = \inf\{ |i| \mid x(i) \neq y(i)\}$ (Does not allow for infinite computations)

Classes of Gilman

f is a cellular automaton map

- Class I: f $\in$ Class I if f is equicontinuous at some $x \in S^Z$
- Class II: f $\in$ Class II if f is almost equicontinuous at some $x \in S^Z$ but f$\notin$ Class I
- Class III: f $\in$ Class III if f is almost expansive

Definitions

f is a cellular automaton map

$N_0 = N \cup \{0\}$

- For $\varepsilon > 0$ and $x \in S^Z$, define $D(x, \varepsilon) = \{y \mid d(f^i(x), f^i(y)) < \varepsilon, \forall i \in N_0\}$ f is *equicontinuous* at x iff $\forall\, \varepsilon\; \exists\, n \in N_0 \ni C_n(x)$ (the open ball of radius 2^{-n}) $\in D(x, \varepsilon)$
- f is *expansive* if $\exists\, \varepsilon > 0 \ni \forall\, x\;\; D(x, \varepsilon) = \{x\}$
- Use the infinite product measure μ of the space S^Z, f is *almost expansive* if $\exists\, \varepsilon > 0 \ni \forall\, x$ $\mu(D(x, \varepsilon)) = 0$

Definition of Almost Equicontinuous

f is *almost equicontinuous* at x iff $\forall\ \varepsilon > 0$

$$\lim_{n \to \infty} \frac{\mu(C_n(x) \cap D(x, \varepsilon))}{\mu(C_n(x))} = 1$$

A Few Notes:

Gilman goes on to define some properties of these classes but does not provide an algorithm for membership.

Class III does not distinguish between countable and uncountable

A New Classification Based on Grossone

- Does not Involve Measure Theory (The Lens of Measure Theory is limited in distinguishing infinite sets).
- Uses Grossone to count the number of elements in the set of elements that follows, under forward iteration, that of a given initial sequence (stays close to under the metric).

The Domain Space of Cellular Automata

Recall S is a finite alphabet, for example S = {*0*, *1*}

And S^Z is the space of all bi-infinite sequences defined on the integers and taking values from S

As usual, let |Y| = the number of elements in the set

<u>Theorem</u>: $|S^Z| = |S|^{2\text{①} + 1}$

Hence we know the number of bi-infinite sequences in the entire space

This is extremely interesting and important since, via Cantor, this space is considered <u>uncountable!</u>

The Metric

Let x and y be two bi-infinite sequences in S^Z

infimum operation

Lower unit

$$x \wedge y = \begin{cases} x & \text{if } x = y \\ * & \text{if } x(0) \neq y(0) \text{ or } x(0) = * \\ x(-n)\ldots x(0)\ldots x(n) & \text{if } x(i) = y(i)\ \forall i \in [-n,n] \text{ and } * \text{ outside} \end{cases}$$

Note, $-n$ can be infinite and equal $-$① $+\ k$ and n can equal ① $-\ k$ for some finite integer $k \geq 0$

Also note in this case, if $k = 0$, then $x = y$.

Hence we can do computations on infinite words

Note, $x \wedge y$ is the largest center stretch where x and y agree

Metric Definition Continued

Define λ as a real valued function taking values in the open interval $(0,1)$ that is $\lambda: S \longrightarrow (0,1)$ But, not infinitesimal

Recall S is a finite alphabet and denote $\lambda_i = \lambda(x(i))$

$$F(x \wedge y) = \begin{cases} 1 & \text{if } x \wedge y = * \\ \prod_{-n}^{n} \lambda_i & \text{if } x \wedge y = * * * x(-n) \ldots x(0) \ldots x(n) * * * \end{cases}$$

And we form the metric:

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ F(x \wedge y) & \text{otherwise} \end{cases}$$

Some Notes

- It is a simple exercise to check this is a metric on the space S^Z
- This metric satisfies the nonarchimedean or ultra metric inequality

$$d(x,y) \leq max\{d(x,z), d(z,y)\}$$

- Great advantage using Grossone: we can use configurations that agree on infinite intervals, and are infinitesimally close to each other.

Open Sets

$C(-n,n,x[-n,n]) = \{x \in S^Z \mid x[-n,n] = w\}$, where $|w| = (2n+1)\}$

A quantity $w = x[-n,n]$ represents a word in the interval $[-n,n]$ and of length $2n+1$.

The disk of radius ε around *x* is

$$C_{[-n,n]}(x) = C(-n,n,x[-n,n])$$

Note n is a natural number $-n \leq 0 \leq n$ and possibly infinite.

It is allowable for n to be an infinite number, for example ① - k, where k is some natural number.

In this case we obtain an open disk of *infinitesimal radius*

More About Open Sets

- To have an open disk of radius ε, the values of λ must first be defined (and assigned).
- Hence, for example, if $\lambda = 1/2 \ \ \forall\, s \in S$, then $C_{[-n,n]}(x)$ is the disk of radius

$$\varepsilon = (1/2)^{(2n+1)}$$

- If n is infinite, then ε is infinitesimal
- Since this metric is nonarchimedean, given any two disks, either one contains the other or they intersect trivially

And Still More

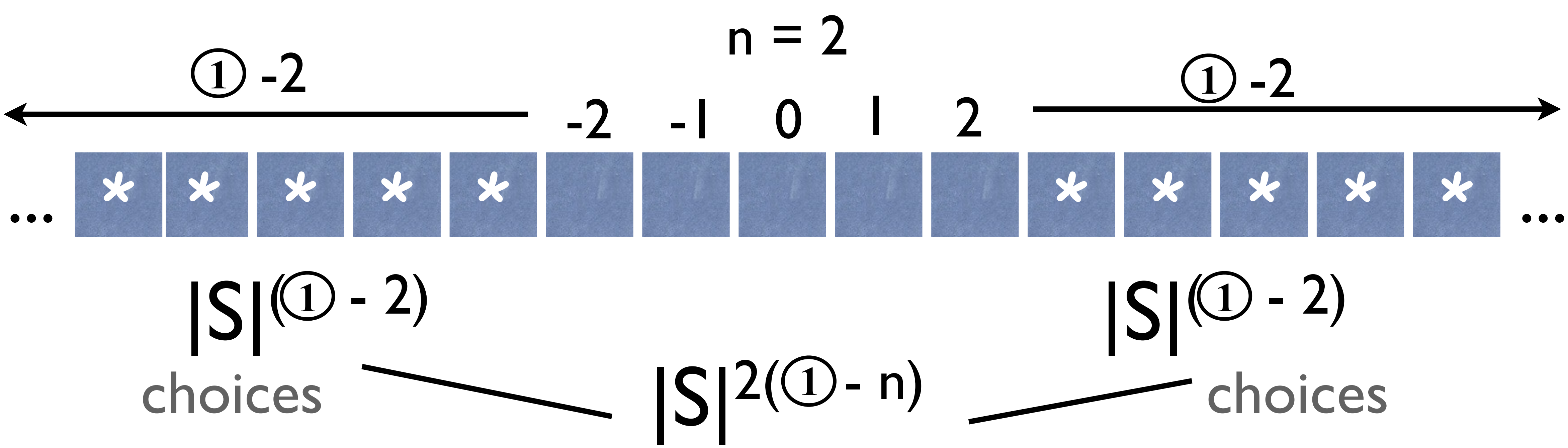
Recall:

<u>Theorem</u>: $|S^{Z}| = |S|^{2① + 1}$

<u>Corollary</u>: The open disk $C_{[-n,n]}(x)$ around x contains

$|S|^{2(① - n)}$ Elements

<u>Proof</u>: Follows directly from the Theorem



Classes of One-Dimensional CA

Understanding the Dynamics

- Necessary to Study the Forward Iterates of the CA function.
- Will allow us to compute *how many* (now possible with Grossone) configurations equal or match that of a given initial configuration.

Let's Do it!

<u>Define</u>:

$B_{m,n}(x) = \{y \mid f^{\,i}(y)[m,n] = f^{\,i}(x)[m,n]\ \forall i \in N_0, m \leq 0 \leq n\}$

Recall: $f^{\,i}(y)[m,n]$ represents the i^{th} iteration of a word, around (not necessarily centered) the 0 position. That is, a center window.

Note that the CA function f is first applied to the entire configuration *x* (or *y*) and then restricted to the interval [*m*,*n*]

Also Note that, due to the *Infinite Unit Axiom*, *m* can equal -①+*k* and, similarly, *n* can equal ①-*k* for some finite integer $k \geq 0$ (not necessarily the same value *k*).

Hence

- The Dynamical Analysis of Cellular Automata presented herein is based on using the *Infinite Unit Axiom* to count the number of elements in the $B_{m,n}$ classes.
- This gives us the precise number of configurations that will equal an initial configuration upon forward evolution (iteration) of the CA function.

Defining the Classes

1. f $\in \mathscr{A}$ if there is a $B_{m,n}(x)$ that contains at least $|S|^{2①-k}$ elements for some finite integer $k \geq 0$

2. f $\in \mathscr{B}$ if there is a $B_{m,n}(x)$ that contains at least $|S|^{\alpha①-k}$ elements for some finite integer $k \geq 0$, $0 < \alpha < 2$ and α not an infinitesimal but f $\notin$ class $\mathscr{A}$.

3. f $\in \mathscr{C}$ otherwise.

Two Resulting Theorems

Theorem A: If there exists a $B_{m,n}(x)$, for cellular automaton f, that contains a disk of non-infinitesimal radius, then f $\in \mathscr{A}$

Theorem B: If f $\in \mathscr{A}$ then there exists a $B_{m,n}(x)$ class that contains a disk (of non-infinitesimal radius).

Examples

$$F(a,b,c) = \begin{cases} 1 & \text{if } a=b=c=1 \\ 0 & \text{otherwise} \end{cases}$$

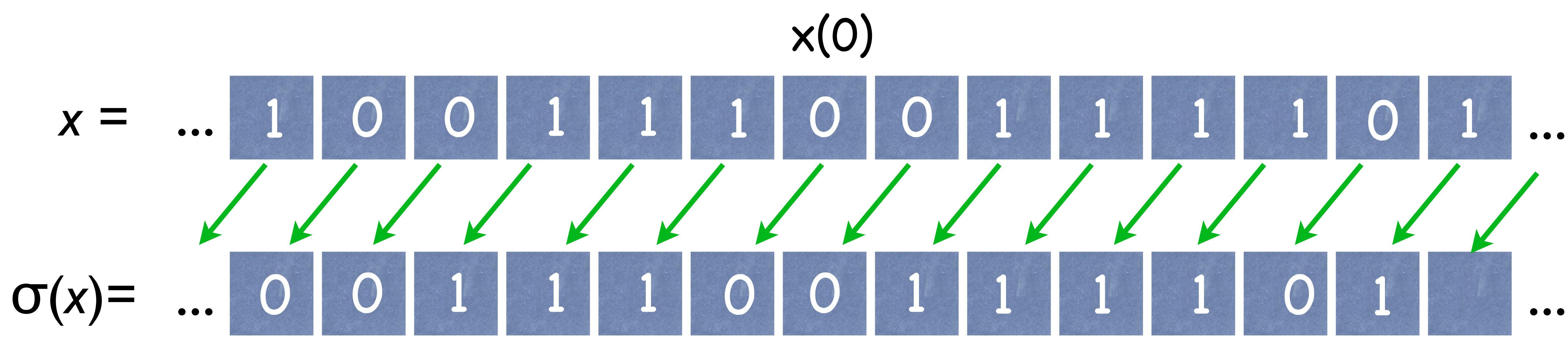
Here, all configurations go to 0 except the configuration of all 1's

By Theorem A, this cellular automaton belongs to class $\mathscr{A}$

The Famous Shift Map

The left shift is a cellular automaton of range 1 and defined by $\sigma(x_i) = x_{i+1}$

This is the automaton that shifts all configurations one unit to the left:



Obviously, all configurations $y \in B_{m,n}(x)$ would have to agree with x to the right and out to ① and at the 0^{th} place

The Shift Map Continued

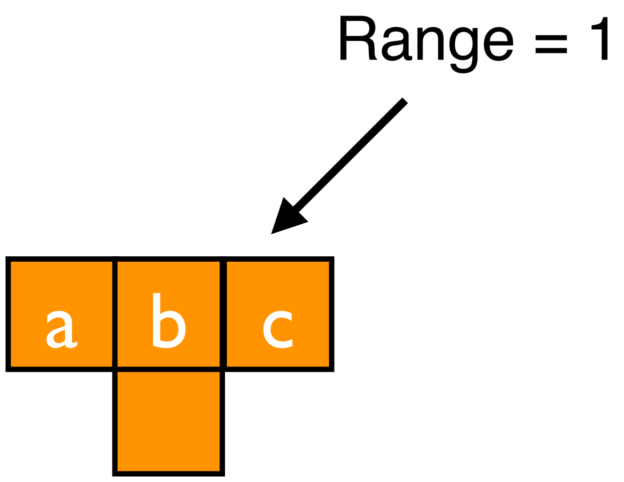
Since all configurations $y \in B_{m,n}(x)$ would have to agree with x to the right and out to ① and at the 0^{th} place:

Hence, there are at most $|S|^{① - k}$ elements, for some $k \geq 0$ in any $B_{m,n}(x)$ and σ must be in Class $\mathscr{B}$.

Modulo Sum Rules

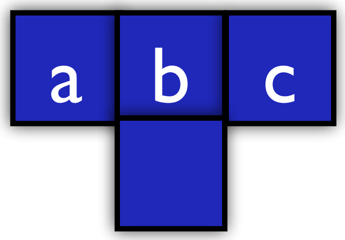
Modulo sum (Additive Cellular Automata)

Some examples for the binary alphabet S = {0,1}:



F(a,b,c) = (b+c) mod 2

F(a,b,c) = (a+c) mod 2



Again, Range = 1

Modulo CA Maps

$$F(a,b,c) = (a+c) \bmod 2$$

								x(0)								
$x =$	...	1	0	0	1	1	1	0	0	1	1	1	1	0	1	...
$f(x)=$	...		1	1	1	0	1	1	1	1	0	0	1	0		...
$f^2(x)=$	...			0	1	0	1	0	0	1	1	1	0			...

The way the CA is defined shows that the automaton generated, by F does not contain an open ball and hence does not belong to class $\mathscr{A}$ Therefore it must belong to class $\mathscr{B} \cup \mathscr{C}$.

And .... Future Work

- Construct an example(s) of a Class 3 cellular automata. Possibly show that the modulo 2 rule does not belong to class $\mathscr{B}$. Another possible example would be a cellular automata that exhibits universal computation, such as Wolfram rule 150[1]
- Develop a finer distinction of classes using Grossone
- Develop an algorithm for membership in each of these classes.

[1] See Wolfram, S., A New Kind of Science