Fractals via supercomputing with infinities and infinitesimals

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An outline of the lecture

This lecture consists of three parts as follows:

1. SECTION 1. What are fractals?
2. SECTION 2. The Sierpinski tetrahedron in higher dimensions.
3. SECTION 3. The Sierpinski curve.

A celebrated example of fractal: the Mandelbrot set

Fractals are geometrical objects that display "self similarity" on all scales (i.e. they exhibit not necessarily exactly the same structure, but the same "type" of structures on all scales).

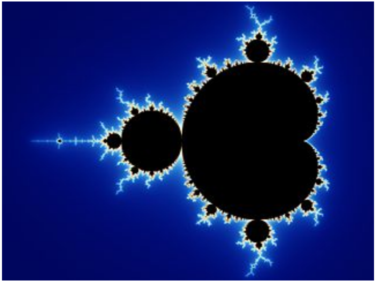

Figure: The *Mandelbrot set M*, which has been labeled "the most complex object in mathematics", is a compact set contained in the closed disc $D_2(0) \subset \mathbb{C}$.

If $f_c(z) := z^2 + c$, then $M := \{ c \in \mathbb{C} \mid \text{the sequence } (f_c(0),\ f_c(f_c(0)),\ f_c(f_c(f_c(0))),\ \ldots) \text{ does not diverge} \}$.

Other examples of fractals. The Sierpinski Carpet.

To obtain the *Sierpinski Carpet* C, we construct a sequence of polygons $\{C_n\}_n$ as follows:

0. We start from the (black) unitary square $C_0 := [0,1] \times [0,1]$.
1. To obtain C_1 we take away from C_0, the WHITE central square $[1/3, 2/3] \times [1/3, 2/3]$. So C_1 consists of 8 small square of side $1/3$.

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2. We divide each of the 8 squares that make up C_1 into 9 smaller squares and remove the central one. Thus, C_2 will be composed of 64 squares of side 1/9.
3. So on for C_3, C_4, etc.

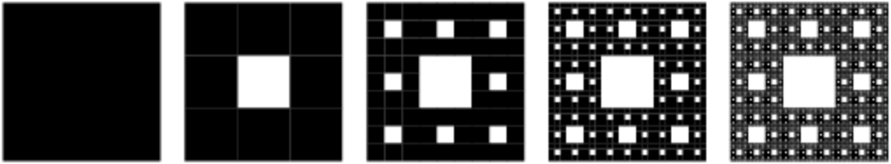

Figure: The first four steps in the construction of the *Sierpinski Carpet* C.

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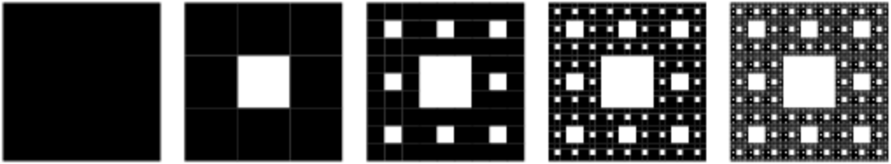

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The Sierpinski Carpet.

Now, the *Sierpinski Carpet* C is defined as the limit

$$C := \lim_{n \to +\infty} C_n,$$

i.e. there exists a unique compact set $C \subset \mathbb{R}^2$ which is the limit of the compact sets C_n. Moreover, note that

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This means that C is a boundary set, i.e. it has empty interior.

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Characteristics and methods of study

- The word “fractal" contains (one of) the most important characteristic of fractals. What is it?
  It was first used by Benoît Mandelbrot in 1975, and it comes from the Latin *fractus* which means “broken" or “fractured" or also “ratio".
  Erroneously, many people think that this term is due to the fractured aspect of fractals, instead Mandelbrot coined it to extend the concept of theoretical **fractional dimensions** to geometric patterns in nature.
- A *fractal dimension* is an index associated to a fractal shape that would measure its complexity by a positive real (usually non integer) number.
- Several types of fractal dimension can be measured theoretically and empirically (more than 20, like *box counting dimension*, *Cech dimension*, *Packing dimension*, etc.), the most important one is the *Hausdorff dimension*. They agree on some classical fractals, but they are not equivalent.

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Characteristics and methods of study

- Mathematicians are in disagreement about how the concept of a fractal should be formally defined. But there are some required key-features on which there is a general agreement, the most important of them is an infinitely self-similarity (distinguished in *exact, quasi, statistical, qualitative self-similarity*) of the object (not necessarily geometric).

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  "*beautiful, damn hard, increasingly useful. That's fractals.*"

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Fractals in nature, techonology and applications

- Fractal features are found in many natural phenomena like
  - snowflakes, crystals, ocean waves, lightning bolts, trees, algae;
  - vegetables as broccoli, pineapple, or animal coloration patterns;
  - coastlines, mountain range, fault lines, river networks;
  - proteins, DNA, blood and pulmonary vessels, and many others.

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Possible applications of Sergeyev's system

Some possible new applications of the Grossone system:

- Fractal geometries.
  On the topic there more than 15 papers by Y.D. Sergeyev himself and other authors.
  Among them the following, on which will be based Section 2 of this lecture.

  C. F. *Evaluating the exact measures of the Sierpinski d-dimensional tetrahedron.* Submitted.

- Space-filling curves.
  The following is the basis for Section 3 of the lecture.

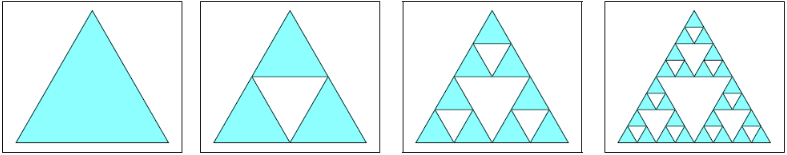
  C. F. *The Sierpinski curve viewed by numerical computations with infinities and infinitesimals.* Applied Mathematics and Computation (2017) (in press).

SECTION 2.
The Sierpinski tetrahedron in higher dimension

Sierpinski Tetrahedron metal sculpture
Breckenridge, Minnesota (USA)

The Sierpinski gasket

- The Sierpinski gasket is defined as the limit of the sequence of polygons $\{\Delta_n^2\}_{n\in\mathbb{N}}$, whose first steps are the following



(a) Δ_0^2 (b) Δ_1^2 (c) Δ_2^2 (d) Δ_3^2

- It has been constructed as a curve by W. Sierpinski one hundred years ago, but it appeared as a decorative pattern many centuries before, for example in Italian medieval art (e.g. the Cosmati mosaics), and in several Roman churches and Basiliche from the 11th century.

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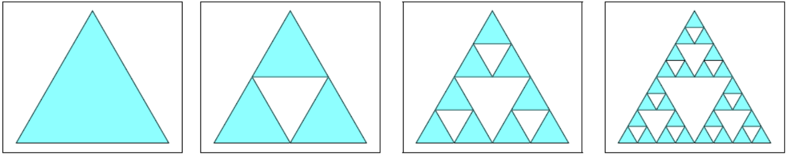

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Sierpinski in medieval Roman basiliche: some examples

(a) Basilica of SS. Giovanni e Paolo, Rome (floor 13th century).

(b) Basilica of San Lorenzo fuori le mura, Rome (floor 13th century).

(c) Basilica of San Clemente, Rome (late 11th century).

(d) Santa Maria Maggiore, Cathedral of Civita Castellana (12th century).

(e) San Lorenzo fuori le mura, Rome (floor 13th century). Sierpinski gasket in six fold circular wedges.

(f) Basilica of Santa Maria in Cosmedin, Rome.

The Sierpinski d-dimensional tetraedron

- The d-dimensional generalization of the Sierpinski gasket starts from Δ_0^d, the *unitary d-simplex* or *unitary d-tetrahedron*.
  The *standard d-simplex* $\mathcal{S}^d$, widespread in many areas of mathematics like singular homology, which is the convex hull of the standard basis $(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$ of $\mathbb{R}^{d+1}$, i.e.

$$\mathcal{S}^d := \left\{ (x_0, \ldots, x_d) \in \mathbb{R}^{d+1} \;\middle|\; \sum_{i=0}^{d} x_i = 1 \text{ and } x_i \geqslant 0 \text{ for all } i = 0, \ldots, d \right\},$$

  differs from Δ_0^d because its edges has length $\sqrt{2}$.

- We recall that
  - the d-volume of a regular d-simplex of edge length l is $\frac{\sqrt{d+1}}{d!\sqrt{2^d}} \cdot l^d$;
  - the number of the k-faces $(0 \leqslant k \leqslant d)$ of a d-simplex is given by the binomial coefficient $\binom{d+1}{k+1}$.

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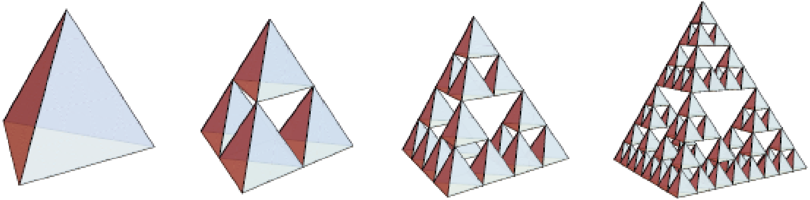
- The sequence $\left\{\Delta_n^d\right\}_n$ in defined in a similar way as for $d = 2$:
  - Δ_1^d is the union of $d+1$ regular tetrahedra of side $1/2$, each one built in a corner of Δ_0^d;
  - then we repeat a copy of Δ_1^d, scaled by $l_{n-1} = (1/2)^{n-1}$, in each small tetrahedron of side $l_{n-1} := (1/2)^{n-1}$ constituting Δ_{n-1}^d,
  - or, equivalently, we replicate a copy of Δ_{n-1}^d, scaled by $l_1 = 1/2$, in each of the $d+1$ tetrahedrons of side $l_1 = 1/2$ composing Δ_1^d.
- The first three steps, in the case $d = 3$, are, for example, the following



(a) $\Delta_0^{(3)}$ (b) $\Delta_1^{(3)}$ (c) $\Delta_2^{(3)}$ (d) $\Delta_3^{(3)}$

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Definition 2.1

For all integers $d \geqslant 2$ and $n \geqslant 0$, let $v_n^{d,d}$ be the d-volume of Δ_n^d. Moreover, if $0 \leqslant k < d$, let $v_n^{d,k}$ be the k-volume of the *k-dimensional elements* lying on the $(d-1)$-dimensional boundary surface of Δ_n^d.

Proposition 2.2

For all $n \geqslant 0$ and $d \geqslant 2$, we have

$$v_n^{d,k} = \begin{cases} \dfrac{\sqrt{k+1}}{k!\sqrt{2^k}} \cdot \dbinom{d+1}{k+1} \cdot \left(\dfrac{d+1}{2^k}\right)^n & \text{if } 1 \leqslant k \leqslant d, \\[2ex] \dfrac{(d+1)^{n+1} + d + 1}{2} & \text{if } k = 0. \end{cases} \tag{1}$$

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If we denote by $\dim(\Delta^d)$ the *fractal dimension* of Δ^d, $d \geqslant 2$, it is simple to prove the following

Theorem (J. Kim and H. Kim, 2001)

For every $d \geqslant 2$

$$\dim(\Delta^d) = \frac{\ln(d+1)}{\ln 2} = \log_2(d+1).$$

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Observations:

(1) The limit (2) is zero, or a finite real number > 0, or $+\infty$.

(2) $v_\infty^{d,k}$ is a nonzero finite number $\Leftrightarrow$ $k \geqslant 2$ and $d = 2^k - 1$ $\Leftrightarrow$
$\Leftrightarrow$ $\dim(\Delta^d)$ is an integer (i.e. in very few special cases).

(3) Almost all $v_\infty^{d,k}$ are zero or $+\infty$. We **can not distinguish**

zeros from zeros, and $+\infty$ **from** $+\infty$,

although **they arise and have completely different meanings** one from another.

(4) More precisely, if we fix $d \geq 2$, the number of occurrences of $+\infty$ [of zero] in the set $\{v_\infty^{d,k} \mid 0 \leqslant k \leqslant d\}$ is equal to $\lceil \log_2(1+d) \rceil$ $[1 - \lfloor \log_2(1+d) \rfloor$, respectively].

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Table 1: Some of the values of the limit (2) in dependence from d and k, until the dimension $d = 8$ and for $d = 15, 32, 63, 127$. In the missing lines the values of $v_\infty^{d,k}$ are obvious.

d	dim (Δ^d)	$v_\infty^{d,0}$	$v_\infty^{d,1}$	$v_\infty^{d,2}$	$v_\infty^{d,3}$	$v_\infty^{d,4}$	$v_\infty^{d,5}$	$v_\infty^{d,6}$	$v_\infty^{d,7}$	$v_\infty^{d,8}$	...
2	$\log_2 3$	$+\infty$	$+\infty$	0	–	–	–	–	–	–	...
3	2	$+\infty$	$+\infty$	$\sqrt{3}$	0	–	–	–	–	–	...
4	$\log_2 5$	$+\infty$	$+\infty$	$+\infty$	0	0	–	–	–	–	...
5	$\log_2 6$	$+\infty$	$+\infty$	$+\infty$	0	0	0	–	–	–	...
6	$\log_2 7$	$+\infty$	$+\infty$	$+\infty$	0	0	0	0	–	–	...
7	3	$+\infty$	$+\infty$	$+\infty$	$\frac{35}{3\sqrt{2}}$	0	0	0	0	–	...
8	$\log_2 9$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	0	0	0	0	0	...
...	...	...	...	...	...	...	...	...	...	...	...
15	4	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$\frac{91\sqrt{5}}{2}$	0	0	0	0	...
...	...	...	...	...	...	...	...	...	...	...	...
31	5	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$\frac{18879\sqrt{3}}{10}$	0	0	0	...
...	...	...	...	...	...	...	...	...	...	...	...
63	6	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$\frac{3235501\sqrt{7}}{30}$	0	0	...
...	...	...	...	...	...	...	...	...	...	...	...
127	7	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$\frac{5957094385}{84}$	0	...
...	...	...	...	...	...	...	...	...	...	...	...

A provocative example from Pirahã culture.

The Pirahã are an indigenous people of the Amazon Rainforest in Brazil. They number 200 or 300 individuals, and their language and culture are very interesting for researchers in linguistics, psychology, etc., because have a number of unusual features.
For example, it is easy to read titles about them like *"The language without numbers"*; but actually, they have (only) the concepts of "one", "two", and **"many" for each quantity** > 2 (and "nothing").

A (provocative) question: Assume that a Pirahã men understood what I said so far; How would he write the entries in the Table 1? Like the ones below?

d	$v_\infty^{d,0}$	$v_\infty^{d,1}$	$v_\infty^{d,2}$	$v_\infty^{d,3}$	$v_\infty^{d,4}$	$v_\infty^{d,5}$	$v_\infty^{d,6}$	$v_\infty^{d,7}$	$v_\infty^{d,8}$	...
6	m	m	m	0	0	0	0	—	—	...
7	m	m	m	m	0	0	0	0	—	...
8	m	m	m	m	0	0	0	0	0	...
...	...	...	...	...	...	...	...	...	...	...
15	m	m	m	m	m	0	0	0	0	...
...	...	...	...	...	...	...	...	...	...	...
31	m	m	m	m	m	m	0	0	0	...
...	...	...	...	...	...	...	...	...	...	...
63	m	m	m	m	m	m	m	0	0	...
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Table 2: Actually, the red m are not so "many" as the blue m !!!

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...	...	...	...	...	...	...	...	...	...	...
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The sequences $\left\{v_n^{d,k}\right\}_n$ adopting the new numerical system

Now it is obvious that, choosing any numerical system with infinities and infinitesimals, we get a richer table than Table 1 (and Table 2!!). Adopting Sergeyev's system, and executing ① steps in the construction of Δ^d, we obtain the following values for the related k-volumes

$$v_{①}^{d,k} = \frac{\sqrt{k+1}}{k!\sqrt{2^k}} \binom{d+1}{k+1} \cdot \left(\frac{d+1}{2^k}\right)^{①} \tag{3}$$

in the case $1 \leqslant k \leqslant d$, and

$$v_{①}^{d,0} = \frac{d+1}{2} \cdot (d+1)^{①} + \frac{d+1}{2} \tag{4}$$

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Table 3: The first explicit values of $v_{\textcircled{1}}^{d,k}$, until the dim. $d = 7$ and for $k \leqslant 4$.

d	$v_{\textcircled{1}}^{d,0}$	$v_{\textcircled{1}}^{d,1}$	$v_{\textcircled{1}}^{d,2}$	$v_{\textcircled{1}}^{d,3}$	$v_{\textcircled{1}}^{d,4}$	...
2	$\frac{3}{2}\cdot 3^{\textcircled{1}} + \frac{3}{2}$	$3\left(\frac{3}{2}\right)^{\textcircled{1}}$	$\frac{\sqrt{3}}{4}\left(\frac{3}{4}\right)^{\textcircled{1}}$	–	–	...
3	$2\cdot 4^{\textcircled{1}} + 2$	$6\cdot 2^{\textcircled{1}}$	$\sqrt{3}$	$\frac{1}{6\sqrt{2}}\left(\frac{1}{2}\right)^{\textcircled{1}}$	–	...
4	$\frac{5}{2}\cdot 5^{\textcircled{1}} + \frac{5}{2}$	$10\left(\frac{5}{2}\right)^{\textcircled{1}}$	$\frac{5\sqrt{3}}{2}\left(\frac{5}{4}\right)^{\textcircled{1}}$	$\frac{5}{6\sqrt{2}}\left(\frac{5}{8}\right)^{\textcircled{1}}$	$\frac{\sqrt{5}}{96}\left(\frac{5}{16}\right)^{\textcircled{1}}$	...
5	$3\cdot 6^{\textcircled{1}} + 3$	$15\cdot 3^{\textcircled{1}}$	$5\sqrt{3}\left(\frac{3}{2}\right)^{\textcircled{1}}$	$\frac{\sqrt{5}}{2\sqrt{2}}\left(\frac{3}{4}\right)^{\textcircled{1}}$	$\frac{\sqrt{5}}{16}\left(\frac{3}{8}\right)^{\textcircled{1}}$	...
6	$\frac{7}{2}\cdot 7^{\textcircled{1}} + \frac{7}{2}$	$2570\left(\frac{7}{2}\right)^{\textcircled{1}}$	$420\sqrt{3}\left(\frac{7}{4}\right)^{\textcircled{1}}$	$\frac{35}{3\sqrt{2}}\left(\frac{7}{8}\right)^{\textcircled{1}}$	$\frac{7\sqrt{5}}{32}\left(\frac{7}{16}\right)^{\textcircled{1}}$	...
7	$4\cdot 8^{\textcircled{1}} + 4$	$28\cdot 4^{\textcircled{1}}$	$14\sqrt{3}\cdot 2^{\textcircled{1}}$	$\frac{35}{3\sqrt{2}}$	$\frac{7\sqrt{5}}{12}\left(\frac{1}{2}\right)^{\textcircled{1}}$	...
...	...	...	...	...	...	...

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We have already seen that many values of $v_\infty^{d,k}$ (i.e. entries of Table 1) coincide among them, being equal to zero or to $+\infty$.
So, the following problem arises spontaneously:

Problem 2.3

The numbers $v_{①}^{d,k}$ expressed in new system, are all distinct?
In other words, are coincident entries in Table 3?

The elements $v_{①}^{d,k}$, which carry with them rich information about their original generating sequence, seem, at first sight, quite different from each other.

To prove that they are all effectively distinct, is not a trivial issue; for instance, when $k, h \geqslant 1$, it is equivalent to the following problem

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Problem 2.4

Find all integer solutions $t, d, h, k \in N$ of the following system

$$\begin{cases} \dfrac{\sqrt{k+1}}{k!\sqrt{2^k}} \cdot \dbinom{d+1}{k+1} = \dfrac{\sqrt{h+1}}{h!\sqrt{2^h}} \cdot \dbinom{t+1}{h+1} \\ \dfrac{d+1}{2^k} = \dfrac{t+1}{2^h} \end{cases} \tag{5}$$

with $t > d \geqslant 2$, $1 \leqslant h \leqslant t$ and $1 \leqslant k \leqslant d$.

To prove the nonexistence of such solutions of a Diophantine system as (5), is a non trivial problem; for example, by using the most powerful scientific computational software available today, like *Mathematica*® 11.0 by Wolfram Research Inc., or many others, it is not possible to obtain any answer except for very small values of t cause the complexity of (5).

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Two equivalence definitions in the new system

Hence we try to solve Problem 2.3 theoretically, and for this purpose we pose the following

Definition

Let α and β be any positive quantities expressed in the new computational system.

(i) α and β are said *of the same order* (in symbols $\mathrm{ord}(\alpha) = \mathrm{ord}(\beta)$, or $\alpha \sim_{\mathrm{ord}} \beta$) if their quotient is finite but not infinitesimal.
In case α, β are both infinite or infinitesimal quantities, they are also called *infinities of the same order* or *infinitesimals of the same order*, respectively.

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Two equivalence definitions in the new system: some examples

Example

(1) Assume that α_i has the form a_{i,n_i} ①^{n_i} $+ a_{i,n_i-1}$ ①$^{n_i-1}$ $+ \ldots + a_{i,l_i}$ ①^{l_i} for $i = 1, 2$, where $a_{i,j_i} \in \mathbb{R}$, $n_i, l_i \in \mathbb{Z}$, $l_{j_i} \leqslant j_i \leqslant n_{j_i}$ for all $i = 1, 2$, and assume that the leading coefficients a_{1,n_1} and a_{2,n_2} are different from zero.
Then $\mathrm{ord}(\alpha_1) = \mathrm{ord}(\alpha_2)$ if and only if $n_1 = n_2$ and in particular they are infinities of the same order if $n_1 = n_2 > 0$ or infinitesimals of the same order if $n_1 = n_2 < 0$. Moreover, α_1 and α_2 are equivalent if and only if $n_1 = n_2$ and $a_{1,n_1} = a_{2,n_2}$.

(2) Assume now that $\beta_i = r_i \cdot b_i$^{①}, where r_i, b_i are nonzero real number, $i = 1, 2$. Then $\mathrm{ord}(\beta_1) = \mathrm{ord}(\alpha_2)$ if and only if $b_1 = b_2$ and they are infinities or infinitesimal depending on whether the absolute value of b_1 and b_2 is greater or less than one. Note moreover that $\beta_1 \sim_{eq} \beta_2$ if and only if $b_1 = b_2$ and $r_1 = r_2$.

(3) If d is any integer $\geqslant 2$, then $(d+1)$$^{①+5}$ has the same order of $v_{①}^{d,\,0} = \frac{d+1}{2} \cdot (d+1)$^{①} $+ \frac{d+1}{2}$, but they are not equivalent; instead $(1/2) \cdot (d+1)$$^{①+1}$ is an infinity equivalent to $v_{①}^{d,\,0}$, but they are not equal numbers because they differ by a finite quantity.

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- The introduced relations are equivalence relations; we want to study their equivalence classes under $\sim_{\mathrm{ord}}$ or $\sim_{eq}$.
- We define, for all $d \geqslant 2$,
  - $V^d := \left\{ v_{①}^{d,k} \;\middle|\; 0 \leqslant k \leqslant d \right\}$,
  - $W^d := \bigcup_{i=2}^{d} V^i$, and
  - $W := \bigcup_{d \geqslant 2} V^d$.
- We denote the equivalence class of $v_{①}^{d,k}$ in W, with respect the relation $\sim_{\mathrm{ord}}$ or $\sim_{eq}$, by

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The main theorem on equivalence classes

- As regards the two equivalence relations $\sim_{\mathrm{ord}}$, $\sim_{eq}$ and the previous defined sets V^d, W^d, W, we can ask
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Theorem

(i) *Let $d \geqslant 2$ and denote by ν_d the number of equivalence classes in the set W^d with respect the equivalence relation $\sim_{\mathrm{ord}}$. Then*

$$\nu_d = \begin{cases} 3 & \text{if } d = 2, \\ \dfrac{3d^2+9d+6}{8} + \dfrac{(-1)^d}{4} \cdot \left\lceil \dfrac{d+1}{2} \right\rceil & \text{if } d \geqslant 3. \end{cases}$$

Moreover, a system of minimal representatives of the classes in W is

$$\begin{aligned} \mathcal{R}_W \quad := \quad & \bigcup \left\{ V^d \mid d \text{ even} \geqslant 2 \text{ or } d = 3 \right\} \\ & \cup \left\{ v_{\textcircled{1}}^{t,\,h} \,\middle|\, t \text{ odd} \geqslant 5 \text{ and } h = 0 \text{ or } \frac{t+3}{2} \leqslant h \leqslant t \right\}, \end{aligned}$$

and for every $v_{\textcircled{1}}^{t,\,h} \in \mathcal{R}_W$, its equivalence class is

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Some consequences of the main theorem

- A trivial consequence of part (ii) of the preceding theorem, is the following

Corollary 2.5

The elements $v_{①}^{d,k}$ are all distinct for every $d \geqslant 2$ and $0 \leqslant k \leqslant d$.

- However, the main theorem tells us much more information than Corollary 2.5; for example
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Hints and examples of new researches and further investigations

- Using Sergeyev's system, there are (important) differences in dependence of the starting point of sequence.
- this means that for each $d \geqslant 2$, there is not only a unique d-dimensional Sierpinski tetrahedron, but we can observe a family of infinitely many d-dimensional Sierpinski tetrahedra.
- We denote by $\Delta^d_{r,n}$, $r, n \in \mathbb{N}_0$, $d \geqslant 2$, the d-dimensional polytope resulting from n iterations starting from Δ^d_r.
- The definition of $v^{d,k}_{r,n}$ and $v^{d,k}_{r,①}$ are clearly the same as those of $v^{d,k}_n$ and $v^{d,k}_{①}$, but starting from Δ^d_r rather than Δ^d_0.

Hints and examples of new researches and further investigations

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- Hence we can write the generalizations of (1) and (4) for all $r \in \mathbb{N}_0$ as follows

$$v^{d,k}_{r,①} = \frac{\sqrt{k+1}}{k!\sqrt{2^k}} \binom{d+1}{k+1} \left(\frac{d+1}{2^k}\right)^r \cdot \left(\frac{d+1}{2^k}\right)^{①}$$

in the case $1 \leqslant k \leqslant d$, and

$$v^{d,0}_{r,①} = \frac{(d+1)^{r+1}}{2} \cdot (d+1)^{①} + \frac{d+1}{2}$$

if $k = 0$.

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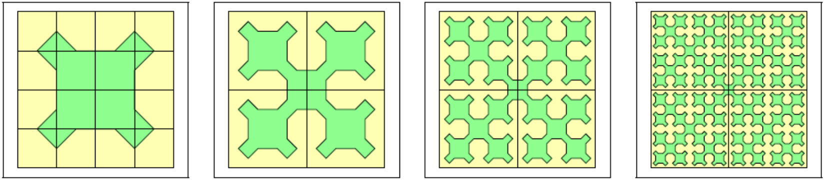
The Sierpinski space-filling curve (only a sketch)

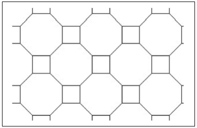
- The Sierpinski curve is one of the most known space-filling curves and one with the highest number of applications.
- There are several different constructions for the Sierpinski curve.
- Our *first construction* is the one obtained by dividing repeatedly a square (with side of length $a > 0$).



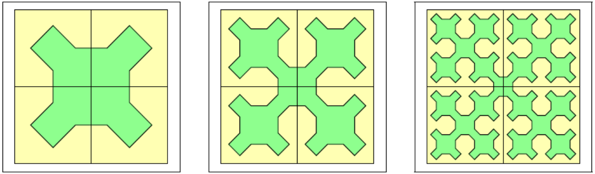
(a) S_0 (b) S_1 (c) S_2 (d) S_3

The Sierpinski space-filling curve

- Another construction comes from the plane tassellation with regular octagons and squares



- This slightly kind of Sierpinski curve, which we call *second construction*, has many applications in electronic, signal theory, computer science, because of its high symmetry.



(a) S'_0 (b) S'_1 (c) S'_2

Some (precise) measures, relative to the fractal curves, expressed in the new system

- For the first construction of the curve:

$$l_{k,①} = \frac{2^k \cdot 4a}{3} \left(1+\sqrt{2}\right) \cdot 2^{①} - \frac{a}{6 \cdot 2^k} \left(2-\sqrt{2}\right) \cdot 2^{-①},$$

$$A_{k,①} = \frac{5}{12}a^2 - \frac{7a^2}{96 \cdot 4^k} \cdot 4^{-①}.$$

- For the second construction:

$$l'_{k,①} = 2^{k+3} a \left(\sqrt{2}-1\right) \cdot 2^{①},$$

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Some power series in the new system arising from comparisons of the curves

- Making comparisons inside the same construction of the curve, or between the two different ways to build them, we obtain some interesting power series expansions expressed in the new system.
- These expansions represent highly precise results and carry much information about the original objects.
- An example from the first curve:

$$\frac{l_{k,①}}{l_{h,①}} = 2^{k-h} + \left(2^{k-2h} - 2^{-k}\right) \cdot \sum_{n \geqslant 1} 2^{-(2n-1)h} \left(\frac{2-\sqrt{2}}{8\left(1+\sqrt{2}\right)}\right)^{n} \cdot \left(4^{-①}\right)^{n}.$$

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Other examples:

- $$\frac{A_{k,①}}{A_{h,①}} = 1 + \left(\frac{1}{4^h} - \frac{1}{4^k}\right) \sum_{n\geqslant 1} \left(\frac{7}{40}\right)^n \frac{1}{4^{(n-1)h}} \cdot \left(4^{-①}\right)^n.$$

- $$\frac{A_{k,①}}{A'_{k,①}} = \frac{35+20\sqrt{2}}{68} + \frac{24\sqrt{2}-213}{68^2} \cdot \sum_{t\geqslant 1} 2^{-(1+2k)t} \left(\frac{1+3\sqrt{2}}{17}\right)^{t-1} \cdot \left(4^{-①}\right)^t.$$

Observations:

- Our series are completely different from standard power series and from the series of classical analysis in general, because, even if the word “infinitesimal” is ubiquitous, it has a different meaning.

- For example, a convergent series $\sum a_n$ is, of course, necessarily originated by an infinitesimal sequence $\{a_n\}_n$, but its elements are usual real or complex numbers and **not infinitesimal objects properly said**.

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Hints for further investigations

- The apparition of grossone-based series gives rise to many questions about their behavior, their properties and consequences, their use and their applications, etc.
- For instance:
  - When does such a series converge, diverge, is irregular?
  - And if it "converges", what converges?
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Main references

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