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# Phenomenon of Immobility in study of convex Semi-infinite Programming problems with finitely representable compact index sets: General approach

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# Outline

# Optimization problems

In general, an optimization problem can be formulated in the form

$$(P): \quad \min_{x} c(x)$$
$$\text{s.t. } f(x,t) \leq 0 \quad \text{for all } t \in T,$$

where
$x \in \mathcal{X}$ is *a decision variable vector;*
function $c(x): \ \mathcal{X} \to \mathbb{R}$ is the *objective function*;
function $f(x,t): \ \mathcal{X} \times \mathbb{R}^p \to \mathbb{R}$ is the *constraint function*,
and the set $T \subset \mathbb{R}^p$ is the *index set.*

Problem $(P)$ is a Nonlinear Programming (NLP) problem if $x \in \mathcal{X} = \mathbb{R}^n$ and the index set $T$ is finite, $T = \{t_1, t_2, ..., t_m\}, \ t_i \in \mathbb{R}^s$ :

## Nonlinear Programming (NLP)

$$\text{NLP:} \quad \min_{x \in \mathbb{R}^n} c(x)$$
$$\text{s.t. } g_i(x) := f(x, t_i) \leq 0, \ i = 1, 2, ..., m.$$

# Semi-Infinite Programming Problems

$$(P): \quad \min_{x} c(x)$$
$$\text{s.t. } f(x,t) \le 0 \ \text{для всех } t \in T.$$

Problem $(P)$ is a Semi-Infinite Programming (SIP) problem if

- the decision variable vector $x$ is finite dimensional: $x \in \mathbb{R}^n$,
- the index set $T$ consists of an infinite number of elements:
  - a countable discrete index set $T = \{t_1, t_2, ...\}$,
  - a compact set in $\mathbb{R}^p$, for example:
    - a compact set in $\mathbb{R}^1$: $T = [t_*, t^*]$;
    - a polyhedral index set in $\mathbb{R}^p$: $T = \{t \in \mathbb{R}^p : h_s' t \le \Delta h_s, \ s \in S\}$;
    - a finitely representable index set in $\mathbb{R}^p$: $T = \{t \in \mathbb{R}^p : g_s(t) \le 0, \ s \in S\}$;
    - an arbitrary compact set in $\mathbb{R}^p$.

A semi-infinite programming problem is called a **Generalized** problem (GSIP) if the index set depends on the decision variable $x$:

$$T(x) = \{t \in \mathbb{R}^s : g_s(t,x) \le 0, \quad s \in S\}.$$

# Motivation. Applications

Due to the numerous theoretical and practical applications, today semi-infinite optimization is a topic of a special interest.


• Mordukhovich B., Nghia T.T.A. *Constraint qualifications and optimality conditions for nonconvex semi-infinite and infinite programs*, Math. Program., Ser. B, No. 139 (2013) pp. 271-300.

• N.Q. Huya, D.S. Kimb. *Stability and augmented Lagrangian duality in nonconvex semi-infinite programming.* Nonlinear Analysis 75 (2012) 163–176

• Fajardo M.D., M.A. López. *Some results about the facial geometry of convex semi-infinite systems.* Optimization. **55**(5-6) (2006) pp. 661–684.

• López M., G. Still. Semi-infinite Programming. *EJOR.* **180** (2) (2007) pp. 491–518.

• Goberna M.A., López M.A. (eds.) *Semi-Infinite Programming: Recent Advances,* Kluwer, Dordrecht, 2001.

• Still G. *Generalized Semi-infinite programming: theory and methods*, European Journal of Operational Research 119 (2)(1999) pp. 301-313.


## Semi-Infinite Programming (SIP) problems naturally arise in

- approximation theory,
- optimal control theory,
- numerical analysis,
- theory of variational inequalities,
- robust optimization.

# Applications


• Hettich R., Kortanek K.O. *Semi-infinite programming: theory, methods and applications*, SIAM Rev. 35, (1993), pp. 380-429.

• Still G. *Generalized Semi-infinite programming: theory and methods*, European Journal of Operational Research 119 (2)(1999) pp. 301-313.

• Jess A., Jongen H.Th., Neralic L., and Stein O. *A semi-infinite programming model in data development analysis*, Optimization 49, (2001), pp. 369-385.

• Weber G.-W. *Generalized semi-infinite optimization: theory and applications in optimal control and discrete optimization*, Optimality Conditions, Generalized Convexity and Duality in Vector Optimization, J. Stat. Manag. Syst. 5, (2002), A.Cambini, L.Martein (eds).

# SIP and NLP problem: connections and differences

## Linear SIP problem

(LSIP):

$$\min_{x\in\mathbb{R}^n} c'x$$

$$\text{s.t. } f(x,t) := a'(t)x + b(t) \leq 0,\ \forall\, t \in T = [t_*, t^*].$$

## Linear programming Problem (LP)

$(LP_m)$ :

$$\min_{x\in\mathbb{R}^n} c'x$$

$$\text{s.t. } a'_i x + b_i \leq 0,\ i = 1, 2, ..., m,$$

$$a_i := a(t_i),\ b_i := b(t_i),\ i = 1, 2, ..., m;$$

$$t_i := t_* + (i-1)\Delta t,\ i = 1, 2, ..., m;\ \ \Delta t := (t^* - t_*)/(m-1).$$

## Problem $(LP_m)$

• The set of feasible solutions is a convex polyhedron.
• Among optimal solutions there exists a basic optimal solution (that is a corner point of the polyhedron).
• If the feasible set is non-empty and the cost function is bounded from below then there exists an optimal solution.
• We do not need Constraint Qualifications to formulate optimality conditions.

## Problem (LSIP)

• The set of feasible solutions is not a convex polyhedron.
• As a rule, among optimal solutions there is no a basic optimal solution.
• The non-emptiness of the feasible set and boundedness of the cost function do not imply the existence of an optimal solution.
• The fulfillment of Constraint Qualifications is essential for formulation of optimality conditions.

In general, $\lim_{m\to\infty} val(LP_m) \neq val(LSIP)$.

Here $val(P)$ denotes the optimal value of the cost function in the corresponding optimization problem $(P)$.

# SIP and NLP problem: connections and differences

Under certain regularity conditions (or *Constraint Qualifications (CQ)*), the original SIP problem

$$\text{SIP:} \qquad \min_{x \in \mathbb{R}^n} c(x)$$
$$\text{s.t. } f(x,t) \le 0, \ \forall\, t \in T = \{t \in \mathbb{R}^s : g_s(t) \le 0, \ s \in S\},$$

can be (locally) reduced to an auxiliary NLP problem

$$\text{NLP:} \qquad \min_{x \in \mathbb{R}^n} c(x)$$
$$\text{s.t. } \bar{f}_i(x) \le 0, \ i = 1, 2, ..., m.$$

This provides the opportunities

- to formulate optimality conditions for SIP problem,
- to develop duality theory,
- to justify numerical methods.

In the SIP literature, there exist two main approaches that allow (under certain QC) to reduce a test of optimality of a given feasible vector $x^0$

in original SIP problem with infinitely many constraints

to a test of optimality of the vector $x^0$

in an auxiliary NLP problem with a finite number of constraints:

Approach I is Discretization Approach,

Approach II is Reduction Approach.

This allows to apply the widespread results of Mathematical Programming.

# Approach I: A Discretization Approach

Infinitely many constraints $f(x,t) \leq 0, t \in T$, are replaced by a finite number of constraints $f(x,t_j) \leq 0, i \in J$, where $\{t_j, j \in J\} \subset T$, $|J| < \infty$.

## Theorem 1

Let (SIP) problem be convex and consistent. Suppose that the following regularity condition holds:

$$\text{for any } n+1 \text{ indices } t_j \in T,\ j = 1, \ldots, n+1, \text{ there exists}$$
$$\text{a vector } \tilde{x} \in \mathbb{R}^n \text{ such that } f(\tilde{x}, t_j) < 0, j = 1, \ldots, n+1.$$

Then a feasible vector $x^0$ is optimal in the original (SIP) problem iff there exists a set

$$\{t_j,\ j \in J_a\} \subset \{t \in T : f(x^0, t) = 0\},\ \ |J_a| \leq n, \tag{1}$$

such that the vector $x^0$ is optimal in discretized NLP problem :

$$SIP_D : \qquad \begin{array}{rc} & \min\limits_{x \in \mathbb{R}^n} c(x), \\ \text{s.t.} & f(x, t_j) \leq 0,\ j \in J_a. \end{array}$$

$$SIP_D: \quad \begin{array}{c} \min\limits_{x\in\mathbb{R}^n} c(x), \\ \text{s.t.} \quad f(x,t_j)\leq 0,\ j\in J_a. \end{array}$$

• For a given feasible $x^0$, it is easy to form problem $SIP_D$.

• Problem $SIP_D$ is a convex programming problem if functions $c(x)$ and $f(x,t)$ are convex w.r.t. $x$.

• To apply this approach one needs the fulfillment of the strong regularity condition.

# Approach II: A Reduction approach

Infinitely many constraints $f(x,t) \le 0, t \in T$, are replaced by a finite number of constraints $f(x, t_i(x)) \le 0, i \in J = J(x)$, where $t_i(x), i \in J$, are optimal solutions of the parametric problem

$$NLP(x): \quad \max_t f(x,t) \quad \text{s.t. } t \in T = \{g_s(t) \le 0, \ s \in S\}.$$

**Theorem 2.** Let $x^0$ be a feasible solution in problem (SIP). Suppose that for any $t \in \{t \in T : f(x^0, t) = 0\}$ the following conditions hold true:
**LICQ**: vectors $\frac{\partial g_s(t)}{\partial t}, s \in S_a(t) \subset S$, are linear independent,
**SSOSOC**: the strong second order sufficient optimality conditions for $t$ in $NLP(x^0)$.
Then $x^0$ is optimal in original (SIP) problem iff it is an optimal solution of the problem

$$SIP_R: \quad \begin{array}{rl} & c(x) \to \min, \\ \text{s.t.} & \bar{f}_i(x) := f(x, t_i(x)) \le 0, \ i \in J(x^0). \end{array}$$

- The problem $SIP_R$ is not convex even if problem (SIP) is convex.
- Problem $SIP_R$ contains functions $t_i(x), i \in J(x^0)$, that are implicitly defined.
- Regularity conditions **LIQC** and **SSOSOC** should be satisfied.

# Literature on SIP theory


- Hettich R., Kortanek K.O. *Semi-infinite programming: theory, methods and applications*, SIAM Rev. 35, (1993), pp. 380-429.

- López M.A., Still G. *Semi-infinite Programming*, European Journal of Operational Research, Vol. 180 (2007) pp. 491-518.

- Stein O., Still G. *On optimality conditions for generalized Semi-Infinite Programming problems*, J.Optim. Theory Appl. 104, N.2, (2000), pp. 443-458.

- Ruckmann J.J., Shapiro A. *Second-order optimality conditions in generalized Semi-Infinite Programming*, Set-Valued Anal. 9, Numbers 1-2, (2001), pp. 169-186.

## The aim of talk:

For **convex** problem $(SIP)$, to prove new optimality conditions that do not require the fulfillment of CQ and are more efficient than the best known optimality conditions.

For this purpose, we use a new reduction approach that

- is based on notions of **immobile indices** and their **immobility orders**,
- more carefully takes into account specific structure and properties of the original SIP problem,
- is more efficient than the traditional reduction approaches.

## Convex SIP Problems with finitely representable index set

$$(\text{SIP}): \quad \min_{x \in \mathbb{R}^n} c(x)$$
$$\text{s.t.} \quad f(x,t) \leq 0, \;\; \forall\, t \in T = \{t \in \mathbb{R}^s : \; g_s(t) \leq 0, \; s \in S\},$$

- $x \in \mathbb{R}^n$ is a decision variable vector;
- $T \subset \mathbb{R}^p$ is a compact set (index set);
- the set $S \subset \mathbb{N}$ is a finite index set;
- functions $c(x) : \mathbb{R}^n \to \mathbb{R}$ and $f(x,t) : \mathbb{R}^n \times T \to \mathbb{R}$ are convex w.r.t. $x \in \mathbb{R}^n$;
- functions $c(x)$, $f(x,t)$ and $g_s(t)$, $s \in S$, are sufficiently smooth.

## Convex SIP Problems with finitely representable index set

$$(\text{SIP}): \quad \min_{x\in\mathbb{R}^n} c(x)$$

$$\text{s.t.} \quad f(x,t) \le 0, \;\; \forall\, t \in T = \{t \in \mathbb{R}^s: \; g_s(t) \le 0, \; s \in S\}.$$

## The set of feasible solutions:

$$X := \{x \in \mathbb{R}^n: \; f(x,t) \le 0, \forall\, t \in T\}.$$

## Definition 1

An index $t \in T$ is said to be *immobile* if $f(x,t) = 0$ for *all* $x \in X$.

## The set of immobile indices:

$$T^* := \{t \in T: \; f(x,t) = 0, \; \forall x \in X\}.$$

# Regularity conditions for SIP problems

- Constraints of problem (SIP) satisfy Slater's condition (**CQ1** ) if

$$\text{there exists a vector } \bar{x} \in \mathbb{R}^n \text{ such that } f(\bar{x}, t) < 0, \forall t \in T.$$

- Constraints of problem (SIP) satisfy Slater type condition (**CQ2**) if

$$\text{for any finite set } \{t_i \in T,\ i = 1, \ldots, n+1\} \text{ there exists}$$
$$\text{a vector } \tilde{x} \in \mathbb{R}^n \text{ such that } f(\tilde{x}, t_i) < 0,\ i = 1, \ldots, n+1.$$

$$\textbf{CQ1} \quad \Longleftrightarrow \quad \textbf{CQ2} \quad \Longleftrightarrow \quad T^* = \emptyset.$$

## Original SIP problem

$$(SIP): \quad \min_{x \in \mathbb{R}^n} c(x)$$
$$\text{s.t.} \quad f(x,t) \le 0 \;\; \forall\, t \in T = \{t \in \mathbb{R}^p : \; g_s(t) \le 0, \; s \in S\}.$$

## The lower level problem

$$LLP(x): \quad \max_t f(x,t) \quad \text{s.t.} \;\; t \in T = \{t \in \mathbb{R}^p : \; g_s(t) \le 0, \; s \in S\}.$$

For $x \in X$:

- any index $t \in T_a(x) := \{t \in T : \; f(x,t) = 0\}$ is an optimal solution of the corresponding problem $LLP(x)$:

$$T_a(x) = sol(LLP(x));$$

- $T^* \subset T_a(x)$.
- Indices $t \in T^*$ are optimal solutions of problem $LLP(x)$ for all $x \in X$:

$$T^* \subset sol(LLP(x)) \text{ for all } x \in X.$$

$sol(P)$ denotes the set of optimal solutions of the corresponding problem $(P)$.

## The lower level problem

$$LLP(x): \quad \max_t f(x,t) \quad \text{s.t.} \quad t \in T = \{t \in \mathbb{R}^p : \; g_s(t) \le 0, \; s \in S\}.$$

For $t \in T$ define:

- *the set of active indices* at $t$ in problem $LLP(x)$:

$$S_a(t) := \{s \in S : \; g_s(t) = 0\}, \tag{2}$$

- *the linearized tangent cone* to the index set $T$ at $t$:

$$L(t) := \{l \in \mathbb{R}^s : \frac{\partial g_s'(t)}{\partial t} l \le 0, \; s \in S_a(t)\}, \tag{3}$$

- *the cone of critical directions* at $t$ in the index set $T$:

$$C(x,t) = \{l \in L(t) : \; \frac{\partial f'(x,t)}{\partial t} l = 0\}. \tag{4}$$

## The lower level problem

$$LLP(x): \quad \max_t f(x,t) \quad \text{s.t. } t \in T = \{t \in \mathbb{R}^p : \ g_s(t) \le 0, \ s \in S\}.$$

For simplicity, suppose the functions $g_s(t), \ s \in S$, are linear

$$g_s(t) := h_s' t - \Delta h_s, \ s \in S,$$

$\Longrightarrow$ the index set $T$ is polyhedron in $\mathbb{R}^p$.

## The first and second order necessary optimality conditions for $t \in T$ in $LLP(x)$

$$\frac{\partial f'(x,t)}{\partial t} l \le 0 \ \ \forall l \in L(t), \quad l' \frac{\partial^2 f(x,t)}{\partial t^2} l \le 0 \ \ \forall l \in C(x,t). \tag{5}$$

- Since $T^* \subset sol(LLP(x))$ for all $x \in X$, then

for $t \in T^*$ inequalities (5) should be satisfied for all $x \in X$.

# Immobility orders

For $\bar{t} \in T^*$,

$$\frac{\partial f'(x,\bar{t})}{\partial t} l \leq 0 \ \ \forall l \in L(\bar{t}), \ l' \frac{\partial^2 f(x,\bar{t})}{\partial t^2} l \leq 0 \ \ \forall l \in C(x,\bar{t}) \ \ \forall \, x \in X. \qquad (6)$$

## Definition 2

Let $\bar{t} \in T^*$ and $l \in L(\bar{t}) \setminus 0$. Define immobility order $q = q(\bar{t}, l)$ of immobile index $\bar{t}$ along direction $l$ as follows

- $q(\bar{t}, l) = 0$ if $\exists \ \bar{x} = x(\bar{t}, l) \in X$ such that $\dfrac{\partial f'(\bar{x},\bar{t})}{\partial t} l < 0$;
- $q(\bar{t}, l) = 1$ if $\dfrac{\partial f'(x,\bar{t})}{\partial t} l = 0, \ \forall x \in X$ and
  $\exists \ \bar{x} = x(\bar{t}, l) \in X$ such that $l' \dfrac{\partial^2 f(\bar{x},\bar{t})}{\partial t^2} l < 0$;
- $q(\bar{t}, l) > 1$ if $\dfrac{\partial f'(x,\bar{t})}{\partial t} l = 0, \ l' \dfrac{\partial^2 f(x,\bar{t})}{\partial t^2} l = 0, \ \forall x \in X.$

## Assumption 1

For all $\bar{t} \in T^*$, the immobility orders along $l \in L(\bar{t}) \setminus 0$ are less or equal to 1:

$$q(\bar{t}, l) \leq 1, \ \ \forall l \in L(\bar{t}) \setminus 0, \ \ \bar{t} \in T^*.$$

Back Back

## Proposition

Suppose that Assumption 1 is fulfilled. Then the set of immobile indices $T^*$ contains a finite number of elements:

$$T^* = \{t_j^*, j \in J_*\} \ \text{ with } 0 \leq |J_*| < \infty.$$

# Parametric presentation of the cone $L(t_j^*)$

For an immobile index $t_j^* \in T^* = \{t_j^*, j \in J_*\}$ consider

**The linearized tangent cone to the index set $T$ at $t_j^*$:**

$$L(j) := L(t_j^*) = \{l \in \mathbb{R}^p : \ h_s' l \le 0, \ s \in S_a(t_j^*)\}.$$

**Parametric presentation of $L(j)$ in terms of the extremal rays:**

$$L(j) = \{l \in \mathbb{R}^p : \ l = \sum_{i \in P(j)} b_i(j)\beta_i + \sum_{i \in I(j)} a_i(j)\alpha_i, \ \alpha_i \ge 0, \ i \in I(j)\},$$

where

$b_i(j) \in \mathbb{R}^p, i \in P(j)$ are the bidirectional extremal rays of $L(j)$,

$a_i(j) \in \mathbb{R}^p, i \in I(j)$, are the unidirectional extremal rays of $L(j)$.

## Algorithms for constructing the extremal rays of a polyhedral cone


• Chernikova N.V. *Algorithm for discovering the set of all the solutions of a linear programming problem*, U.S.S.R. Computational Mathematics and Mathematical Physics 8(6)(1968), pp.282-293.

• Fernandez F., Quinton P. *Extension of Chernikova's algorithm for solving general mixed linear programming problems*, Research Report No 934, (1988), IRISA-Institut de Recherche en Informatique et Systemes Aleatoires (Laboratoire de Rennes), France.


## Algorithm for constructing the set of immobile indices $T^*$ and their immobility orders along the extremal rays


• O.I. Kostyukova, T.V. Tchemisova, S.A. Yermalinskaya, *On the Algorithm of Determination of Immobile Indicies for Convex SIP problems*, IJAMAS-International Journal of Applied Mathematics and Statistics, V. 13, $N^0$ J08, (2008), p.13-33.

• Kostyukova O.I., Tchemisova T.V. *A constructive algorithm of determination of the sets of immobile indices in convex SIP problems with polyhedral index sets*, Technical Report, University of Aveiro, 2012.

Without loss of generality suppose that

- we know the set of the immobile indices

$$T^* = \{t_j^*, j \in J_*\};$$

- for every immobile index $t_j^* \in T^*$ we know:
  - the corresponding extremal rays

$$b_i(j), i \in P(j), \ a_i(j), i \in I(j),$$

    that describe the linearized tangent cone $L(j)$ to the index set $T$ at $t_j^*$;
  - the immobility orders along the extremal rays:

$$q(t_j^*, b_i(j)), i \in P(j); \quad q(t_j^*, a_i(j)), i \in I(j).$$

  - the corresponding index subsets

$$I_*(j) = \{i \in I(j) : \ q(t_j^*, a_i(j)) = 0\}, \quad I_0(j) = I(j) \backslash I_*(j).$$

The set of all $l \in L(j)$ such that $q(t_j^*, l) \geq 1$:

$$C_0(j) := \{l \in \mathbb{R}^p : l = \sum_{i \in P(j)} \beta_i b_i(j) + \sum_{i \in I_0(t)} \alpha_i a_i(j), \alpha_i \geq 0, i \in I_0(j)\} \setminus 0,$$

- $$C_0(j) \subset C(x, t_j^*) \subset L(j) \ \ \forall x \in X.$$

For all $x \in X$, immobile indices $t^*_j$, $j \in J_*$, are solutions of

**The lower level problem**

$$LLP(x): \quad \max_t f(x,t), \quad t \in T = \{t \in \mathbb{R}^p : h'_s t \le \Delta h_s, s \in S\}.$$

Hence, for all $x \in X$, the following relations take place

$$\frac{\partial f'(x,t^*_j)}{\partial t} l \le 0 \ \ \forall l \in L(j), \ l' \frac{\partial^2 f(x,t^*_j)}{\partial t^2} l \le 0 \ \ \forall l \in C(x,t^*_j), \ j \in J_*. \tag{7}$$

$$C_0(j) \subset C(x,t^*_j) \subset L(j) \ \ \forall \ x \in X. \tag{8}$$

Consequently, for all $x \in X$, the following relations should take place

$$\frac{\partial f'(x,t^*_j)}{\partial t} b_i(j) = 0, i \in P(j); \ \frac{\partial f'(x,t^*_j)}{\partial t} a_i(j) = 0, i \in I_0(j); \ \frac{\partial f'(x,t^*_j)}{\partial t} a_i(j) \le 0, i \in I_*(j);$$

$$l' \frac{\partial^2 f(x,t^*_j)}{\partial t^2} l \le 0, \ \forall l \in C_0(j), \ \ j \in J_*. \tag{9}$$

# Theorem 3: Implicit sufficient optimality conditions

A vector $x^0 \in X$ is optimal in problem (SIP), if there exist finite sets

$$\{t_j,\ j \in J_a\} \subset \{t \in T : f(x^0, t) = 0\} \setminus T^*, \tag{10}$$

$$l_{sj} \in \{l \in C_0(j) : l' \frac{\partial^2 f(x^0, t_j^*)}{\partial t^2} l = 0\}, s = 1, ..., m_j,\ j \in J_*, \tag{11}$$

$$\sum_{j \in J_*} m_j + |J_a| \leq n,$$

such that the vector $x^0$ is an optimal solution of auxiliary NLP problem :

$(NLP^*)$:

$$\min_{x \in \mathbb{R}^n} c(x)$$

$$\text{s.t. } f(x, t_j^*) = 0,\ \frac{\partial f'(x, t_j^*)}{\partial t} b_i(j) = 0, i \in P(j);\ \frac{\partial f'(x, t_j^*)}{\partial t} a_i(j) = 0, i \in I_0(j);$$

$$\frac{\partial f'(x, t_j^*)}{\partial t} a_i(j) \leq 0, i \in I_*(j);\ l'_{sj} \frac{\partial^2 f(x, t_j^*)}{\partial t^2} l_{sj} \leq 0,\ s = 1, ..., m_j,\ \ j \in J_*,$$

$$f(x, t_j) \leq 0,\ j \in J_a.$$

Theorem 3 holds true without the convexity assumption.
In convex case, under Assumption 1 statements of Theorem 3 are necessary as well.

# Implicit optimality criterion

Consider a feasible solution $x^0 \in X$ of problem (SIP).

In follows from Theorem 3 that

any necessary / sufficient optimality conditions for $x^0 \in X$
in (finite dimensional ) problem $(NLP^*)$

are

necessary / sufficient optimality conditions for $x^0 \in X$
in the original (infinite dimensional ) problem (SIP).

# Properties of problem $(NLP^*)$

$(NLP^*)$:
$$\min_{x\in\mathbb{R}^n} c(x)$$

$$\text{s.t. } f(x,t_j^*)=0,\ \frac{\partial f'(x,t_j^*)}{\partial t}b_i(j)=0, i\in P(j);\ \frac{\partial f'(x,t_j^*)}{\partial t}a_i(j)=0, i\in I_0(j);$$
$$\frac{\partial f'(x,t_j^*)}{\partial t}a_i(j)\le 0, i\in I_*(j);\ l'_{sj}\frac{\partial^2 f(x,t_j^*)}{\partial t^2}l_{sj}\le 0,\ \ s=1,...,m_j,\ \ j\in J_*,$$
$$f(x,t_j)\le 0,\ j\in J_a.$$

## Proposition 1

Let $Y$ and $X$ be the sets of feasible solutions of problems $(NLP^*)$ and $(SIP)$ respectively. Then $X\subset Y$.

## Proposition 2: Slater type condition

Let Assumption 1 be fulfilled. Then $\exists\ \tilde{x}\in X\subset Y$ such that

$$\frac{\partial f'(\tilde{x},t_j^*)}{\partial t}a_i(j)<0, i\in I_*(j);\ l'\frac{\partial^2 f(\tilde{x},t_j^*)}{\partial t^2}l<0,\ l\in C_0(j),\ j\in J_*,$$
$$f(\tilde{x},t)<0,\ t\in T\setminus T^*.$$

$(NLP^*)$:
$$\min_{x\in\mathbb{R}^n} c(x)$$
$$\text{s.t. } f(x,t_j^*)=0,\ \frac{\partial f'(x,t_j^*)}{\partial t}b_i(j)=0, i\in P(j);\ \frac{\partial f'(x,t_j^*)}{\partial t}a_i(j)=0, i\in I_0(j);$$
$$\frac{\partial f'(x,t_j^*)}{\partial t}a_i(j)\le 0, i\in I_*(j);\ j\in J_*;$$
$$l'_{sj}\frac{\partial^2 f(x,t_j^*)}{\partial t^2}l_{sj}\le 0,\ s=1,...,m_j,\ \ j\in J_*,\ \ f(x,t_j)\le 0,\ j\in J_a.$$

$$Q:=\{x\in\mathbb{R}^n: f(x,t_j^*)=0, \frac{\partial f'(x,t_j^*)}{\partial t}b_i(j)=0, i\in P(j); \frac{\partial f'(x,t_j^*)}{\partial t}a_i(j)=0, i\in I_0(j), j\in J_*\},$$
$$\bar{Q}:=\{x\in Q: \frac{\partial f'(x,t_j^*)}{\partial t}a_i(j)\le 0, i\in I_*(j), j\in J_*\}.$$

## Proposition 3: Convexity of sets and functions

- The set $Q$ is convex.
- Functions $\frac{\partial f'(x,t_j^*)}{\partial t}a_i(j), i\in I_*(j), j\in J_*$, are convex w.r.t. $x$ in $Q$.
- The set $\bar{Q}$ is convex.
- For any $j\in J_*$ and $l\in C_0(j)$, functions $l'\frac{\partial^2 f(x,t_j^*)}{\partial t^2}l$ are convex w.r.t. $x$ in $\bar{Q}$.
- For any $t\in T$, functions $f(x,t)$ are convex w.r.t. $x$ in $\mathbb{R}^n$.

$(NLP^*)$: $$\min_{x\in\mathbb{R}^n} c(x)$$

$$\text{s.t. } f(x,t_j^*)=0,\ \frac{\partial f'(x,t_j^*)}{\partial t}b_i(j)=0, i\in P(j);\ \frac{\partial f'(x,t_j^*)}{\partial t}a_i(j)=0, i\in I_0(j);$$
$$\frac{\partial f'(x,t_j^*)}{\partial t}a_i(j)\leq 0, i\in I_*(j);\ l'_{sj}\frac{\partial^2 f(x,t_j^*)}{\partial t^2}l_{sj}\leq 0,\ s=1,...,m_j,\ \ j\in J_*,$$
$$f(x,t_j)\leq 0,\ j\in J_a.$$

- Problem $(NLP^*)$ is a convex programming problem with "nice" properties:
  - The set of feasible solutions $Y$ of problem $(NLP^*)$ is convex and $X\subset Y$.
  - Functions defining inequality constraints of problem $(NLP^*)$ are convex in $Y$.
  - The problem is formulated in terms of functions that are defined explicitly.
  - There exists $\tilde{x}\in X\subset Y$ such that all inequality constraints of $(NLP^*)$ are inactive.
  - If functions $f(x,t_j^*)$, $x\in\mathbb{R}^n$, $j\in J_*$, are *faithfully convex*, then equality constraints of problem $(NLP^*)$ can be replaced by linear ones: $\exists\ A\in\mathbb{R}^{k\times n}, b\in\mathbb{R}^k, k\leq n$, such that

$$Y=\{x\in\mathbb{R}^n: Ax=b,\ \frac{\partial f'(x,t_j^*)}{\partial t}a_i(j)\leq 0, i\in I_*(j);$$
$$l'_{sj}\frac{\partial^2 f(x,t_j^*)}{\partial t^2}l_{sj}\leq 0,\ s=1,...,m_j,\ \ j\in J_*, f(x,t_j)\leq 0,\ j\in J_a\}.$$

## These properties will allow us

• to prove that under Assumption 1 (that is less restrictive than ones usually used for both discretization and reduction approaches) the following holds true

$$x^0 \in sol(SIP) \iff x^0 \in X,\ x^0 \in sol(NLP^*);$$

• to obtain **new explicit** optimality conditions for original problem (SIP) that are more efficient than known ones,

• to improve numerical procedures.

# Chebyshev approximation

Let be given

- a function $f(t) : \mathbb{R}^p \to \mathbb{R}$,
- a set of approximating functions $a(x, t) : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$, parameterized by $x \in \mathbb{R}^n$,
- a compact set $T \subset \mathbb{R}^p$.

We want to approximate $f(t)$ by functions $a(x, t)$ using the max-norm (Chebyshev-norm) $||f(\cdot)||_\infty = \max_{t \in T} |f(t)|$ on the set $T \subset \mathbb{R}^p$.

Minimizing the approximation error $\mu(x) := ||f(\cdot) - a(x, \cdot)||_\infty$ is a problem that can be equivalently expressed as the SIP problem:

## Equivalent SIP problem

$$\min_{x, \mu} \ \mu$$

$$\text{s. t. } g^\pm(x, t) := \pm(f(t) - a(x, t)) \leq \mu \text{ for all } t \in T.$$

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## NLP problem with uncertainties

$$\text{NLP}(y): \qquad \min_{x\in\mathbb{R}^n} f(x,y) \ \text{ s.t. } g(x,y) \le 0,$$

where $y$ is uncertainty parameter vector, $y \in Y \subset \mathbb{R}^p$.

In a pessimistic model we obtain

## Worst-case (or Robust) Optimization Problem

$$\min_{x\in\mathbb{R}^n} \max_{y\in Y} f(x,y) \ \text{ s.t. } g(x,y) \le 0 \ \ \forall\, y \in Y.$$

## Equivalent SIP Problem

$$\text{SIP} \ : \ \min_{x\in\mathbb{R}^n,\ \mu\in\mathbb{R}} \mu, \ \text{ s.t. } f(x,y) \le \mu, \ g(x,y) \le 0, \ \forall\, y \in Y.$$

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# Semi-Definite Programming Problem

## Linear Semi-Definite Programming Problem

$$\max_{x \in \mathbb{R}^n} c(x), \quad \text{s.t.} \sum_{i=1}^{n} A_i x_i + A_0 \preceq 0,$$

where $A_i \in \mathbb{R}^{p \times p}$, $A_i = A_i'$, $i = 0, 1, ..., n$, are given matrices.

## Equivalent SIP problem:

$$\max_{x \in \mathbb{R}^n} c(x),$$

$$\text{s.t. } f(x, t) := t' \left( \sum_{i=1}^{n} A_i x_i + A_0 \right) t \leq 0, \text{ for all } t \in T = \{t \in \mathbb{R}^p : ||t|| = 1\}.$$

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## Convex SIP Problems with finitely representable index set

$$\text{(SIP)}: \quad \min_{x \in \mathbb{R}^n} c(x)$$
$$\text{s.t.} \quad f(x,t) \le 0, \;\; \forall\, t \in T = \{t \in \mathbb{R}^s : \; g_s(t) \le 0, \;\; s \in S\}.$$

## More general statement

$$\text{(SIP)}: \quad \min_{x \in \mathbb{R}^n} c(x)$$
$$\text{s.t.} \quad f_i(x,t) \le 0, \;\; \forall\, t \in T_i, i = 1, ..., k;$$
$$\bar{f}_m(x) \le 0, \;\; m = 1, ..., m_*,$$

where

$$T_i := \{t \in \mathbb{R}^s : \; g_{si}(t) \le 0, \;\; s \in S(i); \;\; g_{si}(t) = 0, s \in S_*(i)\}, \;\; i = 1, ..., k.$$

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# Proposed Approach

• We do not require the fulfillment of CQ.

• Problem $(NLP^*)$ is convex programming problem.

• Problem $(NLP^*)$ coincides with the problem $(SIP_D)$ if the original SIP problem satisfies CQ.

• In general, problem $(NLP^*)$ differs from $(SIP_D)$ and $(SIP_R)$ problems.

• Problem $(NLP^*)$ is formulated in terms of functions that are explicitly defined.

• It is possible to prove that if a vector $x^0$ (that is feasible in problem (SIP)) is optimal in the problem $(SIP_D)$ or $(SIP_R)$, then it is optimal in problem $(NLP^*)$ as well.

• Under assumptions, that are less restrictive than ones usually used for both discretization and reduction approaches, we can prove that a feasible $x^0$ is optimal in problem (SIP) iff it is optimal in problem $(NLP^*)$.